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THE SOLVABILITY OF THE CONJUGACY PROBLEM FOR CERTAIN HNN GROUPS

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1. **Introduction.** Let B be a free product of finitely generated free groups with infinite cyclic amalgamated subgroups. It is well known that B has a solvable conjugacy problem **[12]**. Suppose B is given by

$$\langle b_1, \cdots, b_n, c_1, \cdots, c_m; R(b_1, \cdots, b_n) = S(c_1, \cdots, c_m)\rangle,$$

and let W and V be words in the generators of B defining nonidentity elements of the same order. Let G be the HNN group in the sense of **[11]** given by

$$\langle a, b_1, \cdots, b_n, c_1, \cdots, c_m; R = S, a^{-1}Wa = V\rangle.$$

Here we show

THEOREM. *If B is residually free and 2-free then G has solvable conjugacy problem.*

Let A consist of those groups B given above such that $m=n$, $S=f(R)$, where $f:\langle b_1, \cdots, b_n;\rangle \to \langle c_1, \cdots, c_n;\rangle$ is an isomorphism and R generates its own centralizer in its factor. From **[5]** and our theorem we obtain

COROLLARY 1. *If B is in A then G has solvable conjugacy problem.*

As a consequence we obtain a result known to a number of workers in this area:

COROLLARY 2. *Let G be a one-relator group given by*

$$\langle a, b_1, \cdots, b_k; a^{-1}P(b_1, \cdots, b_k)a = Q(b_1, \cdots, b_k)\rangle.$$

Then G has solvable conjugacy problem.

Among these groups are the two generator one-relator nonhopfian groups $G(l, m)$ which have been the subject of a great deal of discussion in recent years **[1, 2, 3, 5, 6, 15]**. For concepts and terminology the reader should consult **[14]**, **[16]**.

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2. **The self-conjugacy lemma.** Let B be any group and G be an HNN group given by

$$\text{(I)} \qquad \langle a, B; \operatorname{rel} B, a^{-1}Wa = V\rangle,$$

where W and V are words in the generators of B defining elements of the same order. It follows from Lemma 5 [**16**, p. 18] that if x and y are elements of B which are conjugate in G but not in B then x and y are conjugate in B to powers of W or V and hence in G to powers of W. We will say elements x and y are *power-conjugate* whenever there are integers s, t such that

$$\text{(II)} \qquad x^s = z^{-1}y^tz \neq 1.$$

In particular when $x=y$ and $s\neq t$ in (II) we say x is a *self-conjugate* element. It will be convenient in (II) to say x and y are (s, t) power-conjugate (by z) and similarly the element x is (s, t) self-conjugate (by z). We call the corresponding decision problems the power-conjugacy and self-conjugacy problems. The self-conjugacy problem is studied for $|s|\neq|t|$ in [**3**], [**13**].

We will call B a *Baumslag group* when B is torsion-free, contains no self-conjugate elements, the centralizers of elements are isolated [**8**, p. 16] and B is a U-group [**8**, p. 11]. Among the Baumslag groups are the residually free, 2-free groups ([cf. [**4**], [**5**]] for further discussion). G is said to be a *Baumslag-Solitar* group when G is an HNN group of the type (I) where B is a Baumslag group and W, V define nonidentity elements.

LEMMA 1 (THE SELF-CONJUGACY LEMMA). *Suppose G is a Baumslag-Solitar group. W is (m, n) self-conjugate in G if and only if W and V are (s, t) power-conjugate in B where $m/n=(s/t)^e\neq 1$ and s, t are relatively prime.*

PROOF. Assume W is (m, n) self-conjugate in G by x where x is chosen so that the length of its a-projection [**16**, p. 19] is minimal. The conclusion follows by induction using the results on pinching [**16**, pp. 18–19] and the following properties of power-conjugate elements in Baumslag groups:

(i) if y, z are (k, l) and (k', l') power-conjugate in B then $k/l=k'/l'$, and

(ii) if y and z are (k, l) power-conjugate then (y, z) are $(k/d, l/d)$ power-conjugate where d is the greatest common division of k and l. If W and V are (s, t) power-conjugate in B then for $p>0$, W is (s^p, t^p) self-conjugate in G and Lemma 1 follows.

A simple length-argument yields as in [**21**]:

LEMMA 2. *Let $K=L *_c M$ where L, M are torsion-free, contain no self-conjugate elements and C is infinite cyclic. Let x and y be elements of K of syllable length $p(x)$ and $p(y)$ respectively. Further assume $p(x)$ and $p(y)$ are each $\geqq 2$. We have that x and y are power-conjugate if and only if x and y are*

$(p(y)/d, p(x)/d)$ or $(p(y)/d, -p(x)/d)$ power-conjugate, d the g.c.d. of $p(x)$, $p(y)$.

Hence from Lemma 2, Solitar's theorem [**14**, Theorem 4.6] and a theorem of S. Lipschutz [**12**],

LEMMA 3. *If B is the free product of finitely generated free groups with infinite cyclic amalgamated subgroups then the power-conjugacy problem is solvable in B.*

From Lemmas 1 and 3 we conclude that if B satisfies the hypothesis of the theorem stated in the introduction, then it is solvable whether elements of B are conjugate in G, since we may decide if an element is (l, n) power-conjugate to W or V for some n (cf. the remarks at the beginning of this section).

3. **Equations in groups.** Let G be of the form (I). Let g and h be distinct a-cyclically reduced elements of G which contain a-symbols. It follows from Collin's lemma [**16**, p. 21] that necessary and sufficient conditions that g is conjugate to h are as follows:

(i) there are elements g_0, h_0 where each of g, g_0 and h, h_0 are a-cyclic permutations of the other (cf. [**16**, p. 21] for terminology).

$$g_0 = a^{\varepsilon_1}B_1a^{\varepsilon_2}B_2 \cdots a^{\varepsilon_n}B_n, \qquad h_0 = a^{\varepsilon_1}C_1a^{\varepsilon_2}C_2 \cdots a^{\varepsilon_n}C_n,$$

where $\varepsilon_i = \pm 1$ for $i=1, \cdots, n$ and B_i, C_i are words in the generators of B.

(ii) The following system of equations has a solution: there is a sequence $U_1, \cdots, U_{n+1}$ where each U_i is one of W, V and integers $t_1, \cdots, t_{n+1}$ such that

$$a^{-\varepsilon_i}U_i^{t_i}a^{\varepsilon_i} = \bar{U}_t^{i_i}, \qquad B_i^{-1}\bar{U}_i^{t_i}C_i = U_{i+1}^{t_{i+1}} \qquad i = 1, \cdots, n,$$

where $\varepsilon_1=1$ implies $U_1=W$, $\varepsilon_1=-1$ implies $U_1=V$ and $U_1^{t_1}=U_{n+1}^{t_{n+1}}$.

Rewriting the above equations we obtain a system of the form

$$\text{(III)} \qquad x_i = y_i^{p_i}z_i^{q_i} \qquad i = 1, \cdots, n,$$

where $x_i=B_i^{-1}C_i$, $y_i=B_i^{-1}\bar{U}_iB_i$, $z_i=U_{i+1}$, $p_i=-t_i$, $q_i=t_{i+1}$. Since U_1 is determined there are at most 2^{n-1} distinct sequences $U_1, \cdots, U_{n+1}$, so our problem reduces to solving systems of type (III).

LEMMA 4. *If B is a finitely presented residually free and 2-free group then systems of equations of type* (III) *are solvable.*

PROOF. Since B is residually finite, B has solvable word problem [**10**], [**17**] so we may determine whether y_i and z_i commute. If $[y_i, z_i]\neq 1$ we may produce a free image B/N such that $[y_i, z_a]\neq 1 \bmod N$. Now it follows from Lemma 3 [**18**] that we may decide whether $x_i=y_i^{p_i}z_i^{q_i} \bmod N$

possesses a solution p, q. Moreover, such a solution when it exists is unique and may be constructed so that we need only test to see whether $x_i = y_i^p z_i^q$ in B. If $[y_i, z_i] = 1$ a similar argument suffices. Note y_i, z_i generate a free cyclic group so that when solutions exist they will coincide with the solutions of a linear equation which we can produce. Thus the solvability of a system of type (III) reduces to the solvability of a system of simultaneous linear equations.

Hence we can decide whether g_0, h_0 are conjugate and our theorem is proved.

A systematic treatment of these results using the methods of **[18]**, **[19]**, **[20]** will appear at a latter date.

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