D. Epstein has shown [1] that for quite general groups of homeomorphisms, the commutator subgroup is simple. In particular, let $M$ be a manifold. (In this note, we assume all manifolds are finite dimensional, Hausdorff, class $C^\infty$, and have a countable basis for their topology.) By a $C^r$ mapping (resp. diffeomorphism) we mean a $C^r$ mapping (resp. diffeomorphism) whose $r$th derivative is Lipschitz. Let $\text{Diff}(M, r)$ (resp. $\text{Diff}(M, r^+)$) denote the group of $C^r$ (resp. $C^{r^+}$) diffeomorphisms $h$ of $M$ such that there is an isotropy $H_t$ of $h$ to the identity, and a compact set $K$ such that $H_t(x)=x$ if $x \in M-K$. Epstein showed in [1] that the commutator subgroup of $\text{Diff}(M, r)$ (resp. $\text{Diff}(M, r^+)$) is simple. Thurston announced in [4] that $\text{Diff}(M, \infty)$ is simple. Let $n=\dim M$. In this note we announce the following two results.

**Theorem 1.** If $\infty \geq r \geq n+1$, then $\text{Diff}(M, r^+)$ is simple.

**Theorem 2.** If $\infty \geq r \geq n-1$, then $\text{FT}^r_n$ is $(n+1)$-connected.

Here $\text{FT}^r_n$ denotes Haefliger's classifying space for codimension $n$ foliations of class $C^{r+}$. These two theorems are closely related by results of Thurston (cf. [2], [4]). The case $r=\infty$ of these theorems is due to Thurston [4].

Here we outline a proof of Theorem 1. By Epstein's theorem, it is enough to show $\text{Diff}(M, r^+)$ is equal to its own commutator subgroup. A well-known argument shows that it is enough to prove the latter in the case $M=\mathbb{R}^n$. Let $A>1$. Let $D_0=\{x \in \mathbb{R}^n: -2 \leq x_j \leq 2, 1 \leq j \leq n\}$. For $1 \leq i \leq n$, let

$$D_i = \{x \in \mathbb{R}^n: -2 \leq x_j \leq 2, 1 \leq j < i, -2A \leq x_j \leq 2A, i \leq j \leq n\}.$$  

Let $\alpha \in \text{Diff}(\mathbb{R}^n, \infty)$ be such that $\alpha(x)=Ax$ if $x \in D_0$. Let $\rho$ be a $C^\infty$ real valued function on $\mathbb{R}^n$, with compact support, such that $0 \leq \rho \leq 1$, and $\rho=1$ on $D_1$. Let $\tau_i=\exp(\rho \partial/\partial x_i)$. Then $\tau_i \in \text{Diff}(\mathbb{R}^n, \infty)$, $1 \leq i \leq n$. 

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LEMMA. There exists \( A_0 \) such that the following holds. Let \( A \supseteq A_0 \). Let \( f \) be a \( C^{r+} \) diffeomorphism of \( \mathbb{R}^n \), with support in \( D_0 \), and sufficiently close to the identity (with respect to the \( C^{r+} \) topology). Suppose \( \infty > r \geq n+1 \). Then there exist \( g_0, g_1, \ldots, g_n, \lambda_1, \ldots, \lambda_n \in \text{Diff}(\mathbb{R}^n, r+) \) such that

\[
(1) \quad \alpha f g_n \alpha^{-1} = g_0, \\
(2) \quad \lambda_i g_{i-1} \tau_i \lambda_i^{-1} = g_i \tau_i, \quad 1 \leq i \leq n.
\]

Proof that the lemma implies Theorem 1. It is enough to show that any diffeomorphism such as \( f \) is a product of commutators, since any element of \( \text{Diff}(\mathbb{R}^n, r+) \) is a product of conjugates of such diffeomorphisms. Now if \( u \in \text{Diff}(\mathbb{R}^n, r+) \), let \([u]\) denote its image in the commutator quotient group. From (1), we get \([f][g_n] = [g_0]\), and from (2), we get \([g_{i-1}] = [g_i], 1 \leq i \leq n\). Hence \([f] = 1\). Q.E.D.

Outline of the proof of the lemma in the case \( r < \infty \). Let \( B_0 \) denote the subset of \( \text{Diff}(\mathbb{R}^n, r+) \) consisting of \( g \) with support in \( D_0 \) such that

\[
\sup_{x \neq y} \|D^r g(x) - D^r g(y)\|/\|x - y\| < \delta.
\]

For \( \delta > 0 \) sufficiently small, and \( f \) sufficiently near the identity, there exists a mapping \( \Phi: B_0 \to B_0 \) such that if \( g \in B_0 \) and \( g_0 = \Phi(g) \), then there exist \( g_1, \ldots, g_n, \lambda_1, \ldots, \lambda_n \in \text{Diff}(\mathbb{R}^n, r+) \) such that

\[
(3) \quad \alpha f g_n \alpha^{-1} = g_0, \\
(4) \quad \lambda_i g_{i-1} \tau_i \lambda_i^{-1} = g_i \tau_i, \quad 1 \leq i \leq n.
\]

The mapping \( \Phi \) is continuous with respect to the \( C^r \) topology. Since \( B_0 \) is compact with respect to the \( C^r \) topology, and convex, it follows from the Schauder-Tychonoff fixed point theorem that \( \Phi \) has a fixed point. But such a fixed point provides a solution of the equations (1), (2).

We can only sketch the idea of the construction of \( \Phi \). If \( u: \mathbb{R}^n \to \mathbb{R}^n \) vanishes outside a compact set, we define

\[
\|u\|_{r+} = \sup_{x \neq y} \|D^r u(x) - D^r u(y)\|/\|x - y\|.
\]

Then it is easy to see that if \( g_0 \) is defined by (3), we have

\[
\|g_0 - \text{id}\|_{r+} < A^{-r} \|fg - \text{id}\|_{r+}.
\]

Then we construct \( g_1, \ldots, g_n \) inductively. Supposing \( g_{i-1} \) has been defined, has support in \( D_i \), and is near the identity, we construct \( g_i \) to have support in \( D_{i+1} \), to be near the identity and to satisfy

\[
\|g_i - \text{id}\|_{r+} \leq CA \|g_{i-1} - \text{id}\|_{r+}.
\]

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Here, $C$ is a constant independent of $A$. By taking $A$ sufficiently large, and $f$ sufficiently near the identity in relation to $\delta$, we have that $\Phi$ maps $B_\delta$ into itself, where we define $\Phi(g) = g^n$.

REFERENCES


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