

## A WEIGHTED NORM INEQUALITY FOR FOURIER SERIES

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Let  $Mf(x) = \sup |S_n f(x)|$ , where  $S_n f$  denotes the  $n$ th partial sum of the Fourier series of  $f$ . We will show

$$(1) \quad w \in A_p, p > 1, \text{ implies } \int [Mf]^p w \leq C \int |f|^p w.$$

Recall that a nonnegative weight function  $w \in A_p$ ,  $p > 1$ , if there is a constant  $K$  such that

$$\left( \int_I w \right) \left( \int_I w^{-1/(p-1)} \right)^{p-1} \leq K |I|^p$$

for all intervals  $I$ . The  $A_p$  condition,  $p > 1$ , characterizes all weights  $w$  for which the mapping of  $f$  into the Hardy-Littlewood maximal function of  $f$  is bounded on the weighted  $L^p$  space  $L^p(w)$ . (See Muckenhoupt [6].) This fundamental fact leads to boundedness on  $L^p(w)$  for other operators which can be associated with the Hardy-Littlewood maximal function. For example, the conjugate function and more general singular integrals are of this type. Also,  $\int |S_n f - f|^p w \rightarrow 0$  ( $n \rightarrow \infty$ ) if and only if  $w \in A_p$ ,  $p > 1$ . (See Hunt, Muckenhoupt and Wheeden [5] and Coifman [3].) It follows that the inequality in (1) holds only if  $w \in A_p$ ,  $p > 1$ .

Our proof of (1) follows closely the proof in Coifman [3]. We will prove a Burkholder-Gundy type distribution function inequality which relates the weighted distribution functions of modified versions of  $Mf$  and the Hardy-Littlewood maximal function of  $f$ . (See Burkholder and Gundy [1].) To do this we will use the boundedness of  $M$  on  $L^r$ ,  $r > 1$ , and an extremely useful consequence of the  $A_p$  condition which relates the  $w$ -weighted measure and the Lebesgue measure of certain types of sets. This useful property is closely related to the development of Muckenhoupt [6] and was first explicitly used in connection with a distribution

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function inequality by Fefferman in an unpublished paper. The distribution function inequality implies the  $L^p(w)$  norm of  $Mf$  is majorized by a constant multiple of the  $L^p(w)$  norm of the modified Hardy-Littlewood maximal function of  $f$ . (1) then follows from Muckenhoupt's result on the  $L^p(w)$  boundedness of the Hardy-Littlewood maximal function.

Let

$$H_r f(x) = \sup_{h>0} \left( \frac{1}{2h} \int_{|x-t|<h} |f(t)|^r dt \right)^{1/r}, \quad r \geq 1.$$

Note that  $H_1 f$  is the usual Hardy-Littlewood maximal function and so  $\int [H_1 f]^s w \leq C \int |f|^s w$  if  $w \in A_s, s > 1$ . (See Muckenhoupt [6].) Since  $H_r f = (H_1(|f|^r))^{1/r}$ , it follows that

$$(2) \quad w \in A_{p/r}, r < p, \text{ implies } \int [H_r f]^p w \leq C \int |f|^p w.$$

We will need to use (2) for some  $r > 1$ . This is possible because of the following fundamental result of Muckenhoupt [6]:

$$(3) \quad w \in A_p, p > 1, \text{ implies } w \in A_{p/r} \text{ for some } 1 < r < p.$$

Following Carleson [2], we replace  $Mf$  by

$$M^* f = \sup_n \left| \int_{|x-t|<\pi} e^{-int} f(t) / (x-t) dt \right|.$$

In fact, we will use

$$M^{**} f = \sup_n \sup_{\epsilon > 0} \left| \int_{\epsilon < |x-t| < \pi} e^{-int} f(t) / (x-t) dt \right|.$$

Standard arguments imply

$$(4) \quad Mf \leq C(H_1 f + M^* f) \leq C(H_1 f + M^{**} f) \leq C(H_1 f + H_1(M^* f)).$$

From (4) and (2) with  $r=1$  we see that we may replace  $Mf$  by  $M^{**} f$  in (1). Also, since  $\int [M^* f]^r \leq C \int |f|^r, r > 1$ , (see Hunt [4]) we have

$$(5) \quad r > 1 \text{ implies } \int [M^{**} f]^r \leq C \int |f|^r.$$

Given  $w \in A_p, p > 1$ , choose  $r$  as in (3). We will prove

$$(6) \quad m_w(M^{**} f > 3\lambda, H_r f \leq \gamma\lambda) \leq C(\gamma) m_w(M^{**} f > \lambda),$$

where  $C(\gamma) \rightarrow 0 (\gamma \rightarrow 0)$ . ( $m_w(E) = \int_E w$ .)

Given this weighted distribution function inequality it is easy to complete the proof of (1). From (6) we obtain

$$m_w(M^{**} f > 3\lambda) \leq m_w(H_r f > \lambda\gamma) + C(\gamma) m_w(M^{**} f > \lambda).$$

Hence,

$$p \int_0^\infty \lambda^{p-1} m_w(M^{**}f > 3\lambda) d\lambda \leq p \int_0^\infty \lambda^{p-1} m_w(H_r f > \gamma\lambda) d\lambda + C(\gamma)p \int_0^\infty \lambda^{p-1} m_w(M^{**}f > \lambda) d\lambda$$

and so  $\int [M^{**}f]^p w \leq [\gamma^{-p}/(3^{-p} - C(\gamma))] \int [H_r f]^p w$ . (2) then implies (1).

To prove (6) note that the set  $(M^{**}f > \lambda)$  is open, so  $(M^{**}f > \lambda) = \bigcup I_j$ , where the intervals  $I_j = (\alpha_j, \alpha_j + \delta_j)$  are disjoint and  $M^{**}f(\alpha_j) \leq \lambda$ . It is then sufficient to prove

$$(7) \quad m_w(x \in I_j : M^{**}f > 3\lambda, H_r f \leq \gamma\lambda) \leq C(\gamma)m_w(I_j).$$

We may clearly assume there is a point  $z_j \in I_j$  with  $H_r f(z_j) \leq \gamma\lambda$ .

Let  $\bar{I}_j = (\alpha_j - 2\delta_j, \alpha_j + 2\delta_j)$ ,

$$\begin{aligned} f_1(x) &= f(x), & x \in \bar{I}_j, \\ &= 0, & x \notin \bar{I}_j, \end{aligned} \quad \text{and } f_2 = f - f_1.$$

$m$  will denote Lebesgue measure.

Using (5) we have

$$\begin{aligned} m(M^{**}f_1 > \lambda) &\leq \lambda^{-r} \int [M^{**}f_1]^r \leq C\lambda^{-r} \int |f_1|^r \\ &\leq C\lambda^{-r} [H_r f(z_j)]^r m(I_j) \leq C\gamma^r m(I_j). \end{aligned}$$

For any  $x \in I_j$ ,  $n$  and  $\varepsilon > 0$ ,

$$\left| \int_{\varepsilon < |x-t| < \pi} e^{-int} f_2(t)/(x-t) dt - \int_{\varepsilon < |\alpha_j-t| < \pi} e^{-int} f_2(t)/(x-t) dt \right|$$

is majorized by  $C_0 H_1 f(z_j) \leq C_0 H_r f(z_j) \leq C_0 \gamma\lambda$ . It follows that  $x \in I_j$  implies

$$M^{**}f_2(x) \leq M^{**}f(\alpha_j) + C_0 \gamma\lambda \leq (1 + C_0 \gamma)\lambda,$$

and so

$$M^{**}f(x) \leq M^{**}f_1(x) + M^{**}f_2(x) \leq M^{**}f_1(x) + (1 + C_0 \gamma)\lambda.$$

Hence,  $M^{**}f(x) > 3\lambda$ ,  $x \in I_j$ , implies  $M^{**}f_1(x) > \lambda$  if  $1 + C_0 \gamma < 2$ . Collecting results we obtain

$$(8) \quad m(x \in I_j : M^{**}f(x) > 3\lambda, H_1 f \leq \gamma\lambda) \leq C\gamma^r m(I_j).$$

(7) follows immediately from (8) and the following consequence of the  $A_p$  condition:

(9) If  $w \in A_p$ , any  $p$ , then there are positive constants  $C$  and  $\delta$  such that

for any interval  $I$  and measurable set  $E$ ,  $m(E \cap I) \leq \varepsilon m(I)$  implies  $m_w(E \cap I) \leq C\varepsilon^\delta m_w(I)$ .

To prove (9) we use the fact that  $w \in A_p$ , any  $p$ , implies there is  $s > 1$  and a constant  $C$  such that

$$(10) \quad \left( \int_I w^s \right)^{1/s} \leq C |I|^{(1/s)-1} \int_I w$$

for all intervals  $I$ . (See Muckenhoupt [6].) If  $(1/s) + (1/s') = 1$ , Hölder's inequality and (10) imply

$$\int_{E \cap I} w \leq (m(E \cap I))^{1/s'} \left( \int_I w^s \right)^{1/s} \leq C (m(E \cap I)/m(I))^{1/s'} \int_I w.$$

This gives (9) and completes our proof.

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