A NOTE ON ANOSOV DIFFEOMORPHISMS
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1. Introduction. In this note we shall study a class \( \Gamma \) of diffeomorphisms on a compact \( n \)-dimensional manifold \( M \). The class \( \Gamma \) will include all diffeomorphisms \( F \) with the property that the periodic points of \( F \) are dense in \( M \). Our main theorem will give a characterization of those diffeomorphisms in \( \Gamma \) that are Anosov diffeomorphisms.

2. Statement of results. Let \( F: M \rightarrow M \) be a diffeomorphism on a compact \( n \)-dimensional manifold \( M \) and let \( DF: TM \rightarrow TM \) be the induced derivative mapping on the tangent bundle of \( M \). The mapping \( F \) is said to be an Anosov diffeomorphism if the tangent bundle can be decomposed into a continuous Whitney sum \( TM = E^s + E^u \), such that

(i) \( E^s \) and \( E^u \) are invariant under \( DF \);

(ii) \( DF: E^s \rightarrow E^s \) is contracting, i.e., there exist positive constants \( K \) and \( \lambda, \lambda < 1 \), such that

\[
\| DF^m(v) \| \leq K \lambda^m \| v \|
\]

for all \( v \in E^s \) and \( m \in \mathbb{Z}^+ \);

(iii) \( DF: E^u \rightarrow E^u \) is expanding, i.e., there exist positive constants \( k \) and \( \mu, \mu > 1 \), such that

\[
\| DF^m(v) \| \geq k \mu^m \| v \|
\]

for all \( v \in E^u \) and \( m \in \mathbb{Z}^+ \), cf. [1], [3], and [6].

Since \( DF \) is a homeomorphism, the composed mapping \( DF^m \) is defined for all \( m \in Z \), and this defines a discrete flow on \( TM \). Similarly \( F^m \) is a discrete flow on \( M \), and these flows commute with the natural projection \( p: TM \rightarrow M \). Now let \( \Gamma \) denote the collection of all diffeomorphisms \( F: M \rightarrow M \) such that the union of the minimal sets of the flow \( F^m \) is dense in \( M \). For example, if the periodic points of \( F \) are dense in \( M \), then \( F \in \Gamma \).

For any diffeomorphism \( F: M \rightarrow M \) we define the sets \( \mathcal{B}, \mathcal{S}, \mathcal{U} \) in

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the tangent bundle $TM$ by
\[ \mathcal{B} = \{ v \in TM : \| DF^m(v) \| \text{ is bounded uniformly for } m \in \mathbb{Z} \}, \]
\[ \mathcal{S} = \{ v \in TM : \| DF^m(v) \| \to 0 \text{ as } m \to +\infty \}, \]
\[ \mathcal{U} = \{ v \in TM : \| DF^m(v) \| \to 0 \text{ as } m \to -\infty \}. \]

In the theorem, which we next state, the sets $\mathcal{S}$ and $\mathcal{U}$ will take the role of $E^s$ and $E^u$. However we want to emphasize that the contracting and expanding properties described by (1) and (2) are not included in the definition of $\mathcal{S}$ and $\mathcal{U}$; they follow as one of the consequences of our theorem.

**Theorem.** Let $F : M \to M$ be a diffeomorphism on a compact $n$-dimensional manifold $M$ and assume that $F \in \Gamma$. Then $F$ is an Anosov diffeomorphism if and only if $\mathcal{B} = TM_0$, the zero section of $TM$. Moreover, in this case $\mathcal{S}$ and $\mathcal{U}$ are subbundles of $TM$ which are invariant under the flow $DF^m$, $TM = \mathcal{S} + \mathcal{U}$ (Whitney sum), $DF : \mathcal{S} \to \mathcal{S}$ is contracting, and $DF : \mathcal{U} \to \mathcal{U}$ is expanding.

3. **Outline of proofs.** The proof of this theorem is included in a paper in which we study the general problem of (discrete and continuous) linear flows on vector bundles [4]. The results in this paper also have important applications in the theory of linear differential equations with almost periodic coefficients.

The proof of the necessity of the condition $\mathcal{B} = TM_0$ follows directly from the definition of an Anosov diffeomorphism. The proof of the sufficiency of this condition is accomplished as follows: For $y \in M$ define the fibers
\[ \mathcal{S}(y) = p^{-1}(y) \cap \mathcal{S}, \quad \mathcal{U}(y) = p^{-1}(y) \cap \mathcal{U}, \]
and define
\[ A^+ = \{ v \in \mathcal{S} : \| DF^m(v) \| \leq 1 \text{ for all } m \in \mathbb{Z}^+ \}, \]
\[ A^- = \{ v \in \mathcal{U} : \| DF^{-m}(v) \| \leq 1 \text{ for all } m \in \mathbb{Z}^+ \}. \]

Note that $\mathcal{S}(y)$ and $\mathcal{U}(y)$ are linear subspaces of $T_y M$, the tangent space of $M$ at $y$.

**Step 1.** $A^+$ and $A^-$ are compact sets in $TM$ and there is a $\sigma$, $0 < \sigma < 1$, such that $\{ v \in \mathcal{S} : \| v \| \leq \sigma \} \subseteq A^+$, and $\{ v \in \mathcal{U} : \| v \| \leq \sigma \} \subseteq A^-$. 

**Step 2.** $\mathcal{S}$ and $\mathcal{U}$ are closed sets and $\| DF^m(v) \| \leq \sigma^{-1} \| v \|$ for all $v \in \mathcal{S}$ and $m \in \mathbb{Z}^+$, and $\| DF^{-m}(v) \| \leq \sigma^{-1} \| v \|$ for all $v \in \mathcal{U}$ and $m \in \mathbb{Z}^+$.

**Step 3.** The functions dim $\mathcal{S}(y)$ and dim $\mathcal{U}(y)$ are upper semicontinuous functions of $y$.

**Step 4.** $\mathcal{S}$ and $\mathcal{U}$ are invariant under $DF$ and $DF : \mathcal{S} \to \mathcal{S}$ is contracting and $DF : \mathcal{U} \to \mathcal{U}$ is expanding.
Step 5. If \( y \) belongs to a minimal set for the flow \( F^m \) on \( M \), then

\[
\mathcal{S}(y) + \mathcal{U}(y) = T_yM.
\]

The proof of Step 5 uses essentially a duality argument and reduces to examining properties of the intersection number of singular chains in \( R^n \).

Since (3) holds over a dense set in \( M \) we show, using Steps 2 and 3, that \( \mathcal{S} \) and \( \mathcal{U} \) are subbundles and that \( TM = \mathcal{S} + \mathcal{U} \) (Whitney sum).

**Remarks.** (1) By using a known property of Anosov diffeomorphisms [7], the condition \( \mathcal{B} = T\mathcal{M}_0 \), for \( F \in \Gamma \), then implies that the periodic points of \( F \) form a dense set in \( M \). Also see [3, p. 116].

(2) If one does not assume that the set \( \mathcal{N} = \{ \text{union of the minimal sets of the flow } F^m \} \) is dense in the manifold \( M \), then it still follows, using our general theory [4], that one gets a corresponding splitting of the tangent bundle over the closure \( \overline{\mathcal{N}} \).

We show in [5] that this splitting can be extended to all of \( M \) provided the dimension of the fiber \( \mathcal{S}(y) \) is the same over every minimal set. This fact has also been discovered independently by R. Mañé Ramirez [2] by different techniques.

**References**


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