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A NOTE ON ANOSOV DIFFEOMORPHISMS

BY ROBERT J. SACKER[1] AND GEORGE R. SELL[2]

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1. **Introduction.** In this note we shall study a class Γ of diffeomorphisms on a compact n-dimensional manifold M. The class Γ will include all diffeomorphisms F with the property that the periodic points of F are dense in M. Our main theorem will give a characterization of those diffeomorphisms in Γ that are Anosov diffeomorphisms.

2. **Statement of results.** Let $F:M\to M$ be a diffeomorphism on a compact n-dimensional manifold M and let $DF:TM\to TM$ be the induced derivative mapping on the tangent bundle of M. The mapping F is said to be an *Anosov diffeomorphism* if the tangent bundle can be decomposed into a continuous Whitney sum $TM=E^s+E^u$, such that

(i) E^s and E^u are invariant under DF;

(ii) $DF:E^s\to E^s$ is contracting, i.e., there exist positive constants K and λ, $\lambda<1$, such that

$$\|DF^m(v)\| \leqq K\lambda^m\|v\| \tag{1}$$

for all $v\in E^s$ and $m\in Z^+$;

(iii) $DF:E^u\to E^u$ is expanding, i.e., there exist positive constants k and μ, $\mu>1$, such that

$$\|DF^m(v)\| \geqq k\mu^m\|v\| \tag{2}$$

for all $v\in E^u$ and $m\in Z^+$, cf. [**1**], [**3**], and [**6**].

Since DF is a homeomorphism, the composed mapping DF^m is defined for all $m\in Z$, and this defines a discrete flow on TM. Similarly F^m is a discrete flow on M, and these flows commute with the natural projection $p:TM\to M$. Now let Γ denote the collection of all diffeomorphisms $F:M\to M$ such that the union of the minimal sets of the flow F^m is dense in M. For example, if the periodic points of F are dense in M, then $F\in\Gamma$.

For any diffeomorphism $F:M\to M$ we define the sets $\mathscr{B}$, $\mathscr{S}$, $\mathscr{U}$ in

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the tangent bundle TM by

$$\mathscr{B} = \{v \in TM : \|DF^m(v)\| \text{ is bounded uniformly for } m \in Z\},$$
$$\mathscr{S} = \{v \in TM : \|DF^m(v)\| \to 0 \text{ as } m \to +\infty\},$$
$$\mathscr{U} = \{v \in TM : \|DF^m(v)\| \to 0 \text{ as } m \to -\infty\}.$$

In the theorem, which we next state, the sets $\mathscr{S}$ and $\mathscr{U}$ will take the role of E^s and E^u. However we want to emphasize that the contracting and expanding properties described by (1) and (2) are *not* included in the definition of $\mathscr{S}$ and $\mathscr{U}$; they follow as one of the consequences of our theorem.

THEOREM. *Let $F: M \to M$ be a diffeomorphism on a compact n-dimensional manifold M and assume that $F \in \Gamma$. Then F is an Anosov diffeomorphism if and only if $\mathscr{B} = TM_0$, the zero section of TM. Moreover, in this case $\mathscr{S}$ and $\mathscr{U}$ are subbundles of TM which are invariant under the flow DF^m, $TM = \mathscr{S} + \mathscr{U}$ (Whitney sum), $DF: \mathscr{S} \to \mathscr{S}$ is contracting, and $DF: \mathscr{U} \to \mathscr{U}$ is expanding.*

3. **Outline of proofs.** The proof of this theorem is included in a paper in which we study the general problem of (discrete and continuous) linear flows on vector bundles **[4]**. The results in this paper also have important applications in the theory of linear differential equations with almost periodic coefficients.

The proof of the necessity of the condition $\mathscr{B} = TM_0$ follows directly from the definition of an Anosov diffeomorphism. The proof of the sufficiency of this condition is accomplished as follows: For $y \in M$ define the fibers

$$\mathscr{S}(y) = p^{-1}(y) \cap \mathscr{S}, \qquad \mathscr{U}(y) = p^{-1}(y) \cap \mathscr{U},$$

and define

$$A^+ = \{v \in \mathscr{S} : \|DF^m(v)\| \leqq 1 \text{ for all } m \in Z^+\},$$
$$A^- = \{v \in \mathscr{U} : \|DF^{-m}(v)\| \leqq 1 \text{ for all } m \in Z^+\}.$$

Note that $\mathscr{S}(y)$ and $\mathscr{U}(y)$ are linear subspaces of T_yM, the tangent space of M at y.

Step 1. A^+ and A^- are compact sets in TM and there is a σ, $0 < \sigma < 1$, such that $\{v \in \mathscr{S} : \|v\| \leqq \sigma\} \subseteq A^+$, and $\{v \in \mathscr{U} : \|v\| \leqq \sigma\} \subseteq A^-$.

Step 2. $\mathscr{S}$ and $\mathscr{U}$ are closed sets and $\|DF^m(v)\| \leqq \sigma^{-1}\|v\|$ for all $v \in \mathscr{S}$ and $m \in Z^+$, and $\|DF^{-m}(v)\| \leqq \sigma^{-1}\|v\|$ for all $v \in \mathscr{U}$ and $m \in Z^+$.

Step 3. The functions dim $\mathscr{S}(y)$ and dim $\mathscr{U}(y)$ are upper semicontinuous functions of y.

Step 4. $\mathscr{S}$ and $\mathscr{U}$ are invariant under DF and $DF: \mathscr{S} \to \mathscr{S}$ is contracting and $DF: \mathscr{U} \to \mathscr{U}$ is expanding.

Step 5. If y belongs to a minimal set for the flow F^m on M, then

$$\mathscr{S}(y) + \mathscr{U}(y) = T_y M. \tag{3}$$

The proof of Step 5 uses essentially a duality argument and reduces to examining properties of the intersection number of singular chains in R^n.

Since (3) holds over a dense set in M we show, using Steps 2 and 3, that $\mathscr{S}$ and $\mathscr{U}$ are subbundles and that $TM = \mathscr{S} + \mathscr{U}$ (Whitney sum).

REMARKS. (1) By using a known property of Anosov diffeomorphisms [7], the condition $\mathscr{B} = TM_0$, for $F \in \Gamma$, then implies that the periodic points of F form a dense set in M. Also see [**3**, p. 116].

(2) If one does not assume that the set $N = \{$union of the minimal sets of the flow $F^m\}$ is dense in the manifold M, then it still follows, using our general theory [**4**], that one gets a corresponding splitting of the tangent bundle over the closure $\bar{N}$.

We show in [**5**] that this splitting can be extended to all of M provided the dimension of the fiber $\mathscr{S}(y)$ is the same over every minimal set. This fact has also been discovered independently by R. Mañé Ramirez [**2**] by different techniques.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455