0. The main results. This note is an announcement of the results below, whose proofs will appear separately [7].

Main Theorem. Let $G$ be a linearly reductive affine linear algebraic group over a field $K$ of arbitrary characteristic acting $K$-rationally on a regular Noetherian $K$-algebra $S$. Then the ring of invariants $R = S^G$ is Cohen-Macaulay.

Theorem. If $S$ is a regular Noetherian ring of prime characteristic $p > 0$, and $R$ is a pure subring of $S$ (i.e., for every $R$-module $M$, $M \rightarrow M \otimes_R S$ is injective), e.g., if $R$ is a direct summand of $S$ as $R$-modules, then $R$ is Cohen-Macaulay.

The proofs utilize results of interest in their own right:

Proposition A. Let $L$ be a field, $y_0, \ldots, y_m$ indeterminates over $L$, $S = L[y_0, \ldots, y_m]$, and $Y = \text{Proj}(S) = \mathbb{P}^m_L$. Let $K$ be a subfield of $L$, and let $R$ be a finitely generated graded $K$-algebra with $R_0 = K$. Let $h: R \rightarrow S$ be a $K$-homomorphism which multiplies degrees by $d$. Let $P$ be the irrelevant maximal ideal of $R$, and let $X = \text{Proj}(R)$. Let $U = Y - V(h(P)S)$. Let $\varphi = h^*$ be the induced $K$-morphism from the quasi-projective $L$-variety $U$ to the projective $K$-scheme $X$. Then $\varphi_*^i: H^i(X, \mathcal{O}_X) \rightarrow H^i(U, \mathcal{O}_U)$ is zero for $i \geq 1$.

Proposition A'. Let $(R, P)$ be a local ring of prime characteristic $p > 0$ and let $h$ be a homomorphism of $R$ into a regular Noetherian domain $S$. Suppose that for a certain $i$ the local cohomology module $H^i_P(R)$ has finite length. Then if $i \neq 0$ or $h(P) \neq 0$, the induced homomorphism $H^i_P(R) \rightarrow H^i_P(S)$ is zero.

1. Applications and corollaries. We note that the Main Theorem is stronger than the prior conjectures [2, §0] or [3, p. 56], where $S$ was
assumed to be a polynomial ring over \( K \) and the action was assumed to preserve the grading. (This issue was first raised in [5] and [6].)

Second, we observe that the main result of [2] (where \( G \) was \( \text{GL}(1, K)^m \)), and, in characteristic 0, the main results of [6], the roughly equivalent papers [3], [10], [11] (dealing with the Cohen-Macaulay property for Schubert varieties), and the thesis [9] all follow at once from the Main Theorem here.

Third, we note that the Main Theorem implies that Serre-Grothendieck duality will hold in a useful form (i.e. the pairing will be nonsingular: cf. [1, pp. 5 and 6]) for many orbit spaces of actions of linearly reductive groups on nonsingular varieties.

Fourth, we observe some ideal-theoretic corollaries. Let \( S = K[y_0, \ldots, y_m] \), a polynomial ring, and let \( G \) act so as to preserve degrees. Then \( R = S^G \) will be generated over \( K \) by finitely many forms of \( S \), and we can write \( R = K[z_0, \ldots, z_t]/I \). Here, we assume that \( z_0, \ldots, z_t \) map to generating forms of \( R \), and we grade \( T = K[z_0, \ldots, z_t] \) so that the \( K \)-homomorphism preserves degrees. Then \( I \) will be a homogeneous ideal of \( T \), and is the solution to the “second main problem of invariant theory” (cf. [14, Chapter II, C]) for this particular representation. In this situation the assertion that \( R (\cong T/I) \) is Cohen-Macaulay is equivalent to the assertion that \( I \) is perfect, i.e. \( \text{pd}_T T/I = \text{grade} I \). Then \( \mathcal{H} \) be a graded \( T \)-free minimal resolution of \( T/I \). Then from the Main Theorem and, for example, Theorem 3 and Corollary 1.2 of [4], we have

**Corollary 1.** With notation as above, so that \( S^G \cong T/I \), the length of \( \mathcal{H} \) is \( g = \text{grade} I = \text{height} I \), and \( \mathcal{H} \) is grade-sensitive. That is, if \( u_0, \ldots, u_t \) are elements of a Noetherian \( K \)-algebra \( B \), and we make \( B \) into a \( T \)-algebra by means of the homomorphism \( h \) which takes \( z_i \) to \( u_i \), \( 0 \leq i \leq t \), then if \( J = h(I)B \) and \( E \) is any \( B \)-module of finite type such that \( JE \neq E \), then the grade of \( J \) on \( E \) is the number of vanishing homology groups, counting from the left, of the complex \( \mathcal{H} \otimes_T E \). In particular, if the grade of \( J \) on \( E \) is equal to \( g \), then \( \mathcal{H} \otimes_T E \) is acyclic.

**Corollary 2.** With notation as in Corollary 1, let \( E = B \). Then every minimal prime of \( J = h(I)B \) has height at most \( g \), and if the grade of \( J \) is as large as possible, i.e. \( g \), then \( J \) is perfect (\( \mathcal{H} \otimes_T B \) is acyclic and gives a resolution of length \( g \)) and hence all the associated primes of \( J \) have grade \( g \). If \( J \) has grade \( g \) and \( B \) is Cohen-Macaulay, then the associated primes of \( J \) all have height \( g \) and \( B/J \) is again Cohen-Macaulay.

We also note

**Corollary 3.** If \( K \) has characteristic 0 and \( G \) is semisimple and acts on \( S = K[y_0, \ldots, y_m] \) so as to preserve the grading, then \( R = S^G \) is a Cohen-Macaulay UFD and, hence, Gorenstein.
3. Remarks on the proof of the Main Theorem. When $G$ is linearly reductive there is an $R$-module retraction (the Reynolds operator) of $S$ onto $R=S^G$. This fact is crucial in the proof of the Main Theorem. The proof goes roughly like this: First, we reduce to the case where $G$ is connected and then the theorem is stated slightly more generally—$S$ is only assumed regular at $G$-invariant primes. Utilizing devices involving associated graded rings and generalized Rees rings we make a reduction to a sort of “minimal” graded case: $S$ is the symmetric algebra of a projective module $E$ over a domain $B$ (where $G$ acts on $B$, $E$ and $B$ is a $K$-algebra) and $B$ has no $G$-invariant ideals except 0, $B$. $R=S^G$ is a finitely generated graded algebra over a field, $B^G$, and $R_\mathfrak{p}$ is Cohen-Macaulay except possibly when $\mathfrak{p}=\mathfrak{p}$, the irrelevant maximal ideal (this last condition is what we meant by “minimal”). Let $L$ be the field of fractions of $B$. Then $L \otimes_B S=L[y_0, \ldots, y_m]$ is a polynomial ring. $R$ is a direct summand of $S$ (via the Reynolds operator) and this turns out to imply that $R$ is pure in $L \otimes_B S$. Because of this purity, the maps described in Proposition A, which are zero by Proposition A, are also injective, and one finds that $H^i(X, \mathcal{O}_X)=0$, $i \geq 1$, where $X=\text{Proj}(R)$. [We note that Proposition A itself is proved by a reduction to characteristic $p$.] By “minimality” the local rings of $X$ are Cohen-Macaulay and one can use Serre-Grothendieck duality to show that $R^{(d)}=\sum \oplus_{n \geq 0} R_{nd}$ is Cohen-Macaulay for all large $d$. In the final stages of the proof, we show that $R$ itself is Cohen-Macaulay by reducing to characteristic $p$ a second time. (An important point is that in characteristic $p$ we can take $d=p^e$ for large $e$ and then embed $R \hookrightarrow R^{(d)}$ by using the Frobenius.) A key technical lemma which we use repeatedly in the reductions to characteristic $p$ and which generalizes the usual statements about generic flatness is

**Lemma.** Let $A$ be a Noetherian domain, $R$ an $A$-algebra of finite type, $S$ an $R$-algebra of finite type, $E$ an $S$-module of finite type, and $M$ an $R$-submodule of $E$ of finite type. Then there is an $a \in A-\{0\}$ such that $E_a/M_a$ is $A_a$-free.

The proof of the characteristic $p$ Theorem is easier and uses local cohomology analogues of the arguments in the proof of the Main Theorem.


**Remark 1.** The regularity of $S$ is essential in the statement of the Main Theorem. There are counterexamples when $G=\text{GL}(1, K)$ and $S$ is a graded Cohen-Macaulay Gorenstein UFD. But the regularity of $S$ is used in an apparently rather nongeometric way: it is only used to show that the Frobenius in a certain auxiliary ring, after reducing to characteristic $p>0$, is flat (cf. [8]).
Remark 2. The fact that reduction to characteristic $p$ seems to be essential in the proof of the Main Theorem is odd, because the Main Theorem is primarily a characteristic 0 theorem. There are very few linearly reductive groups in characteristic $p > 0$. See [12]. We note that in [13] techniques related to ours are used to settle a number of other questions.

References