BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

ACTIONS OF REDUCTIVE GROUPS ON REGULAR RINGS AND COHEN-MACAULAY RINGS

BY MELVIN HOCHSTER AND JOEL L. ROBERTS[1]

Communicated by Dock S. Rim, August 30, 1973

0. **The main results.** This note is an announcement of the results below, whose proofs will appear separately [7].

MAIN THEOREM. *Let G be a linearly reductive affine linear algebraic group over a field K of arbitrary characteristic acting K-rationally on a regular Noetherian K-algebra S. Then the ring of invariants $R=S^G$ is Cohen-Macaulay.*

THEOREM. *If S is a regular Noetherian ring of prime characteristic $p>0$, and R is a pure subring of S (i.e. for every R-module M, $M\to M\otimes_R S$ is injective), e.g. if R is a direct summand of S as R-modules, then R is Cohen-Macaulay.*

The proofs utilize results of interest in their own right:

PROPOSITION A. *Let L be a field, $y_0, \cdots, y_m$ indeterminates over L, $S=L[y_0, \cdots, y_m]$, and $Y=\mathrm{Proj}(S)=\boldsymbol{P}_L^m$. Let K be a subfield of L, and let R be a finitely generated graded K-algebra with $R_0=K$. Let $h:R\to S$ be a K-homomorphism which multiplies degrees by d. Let P be the irrelevant maximal ideal of R, and let $X=\mathrm{Proj}(R)$. Let $U=Y-V(h(P)S)$. Let $\varphi=h^*$ be the induced K-morphism from the quasi-projective L-variety U to the projective K-scheme X. Then $\varphi_i^*:H^i(X, \mathcal{O}_X)\to H^i(U, \mathcal{O}_U)$ is zero for $i\geqq 1$.*

PROPOSITION A′. *Let (R, P) be a local ring of prime characteristic $p>0$ and let h be a homomorphism of R into a regular Noetherian domain S. Suppose that for a certain i the local cohomology module $H_P^i(R)$ has finite length. Then if $i\neq 0$ or $h(P)\neq 0$, the induced homomorphism $H_P^i(R)\to H_{PS}^i(S)$ is zero.*

1. **Applications and corollaries.** We note that the Main Theorem is stronger than the prior conjectures [**2**, §0] or [**3**, p. 56], where S was

AMS (MOS) subject classifications (1970). Primary 14M05, 20G05, 14B15, 13D05; Secondary 13H05, 13H10.

[1] Both authors were supported in part by NSF grant GP-37689.

assumed to be a polynomial ring over K and the action was assumed to preserve the grading. (This issue was first raised in [**5**] and [**6**].)

Second, we observe that the main result of [**2**] (where G was $\mathrm{GL}(1, K)^m$), and, in characteristic 0, the main results of [**6**], the roughly equivalent papers [**3**], [**10**], [**11**] (dealing with the Cohen-Macaulay property for Schubert varieties), and the thesis [**9**] all follow at once from the Main Theorem here.

Third, we note that the Main Theorem implies that Serre-Grothendieck duality will hold in a useful form (i.e. the pairing will be nonsingular: cf. [**1**, pp. 5 and 6]) for many orbit spaces of actions of linearly reductive groups on nonsingular varieties.

Fourth, we observe some ideal-theoretic corollaries. Let $S=[y_0, \cdots, y_m]$, a polynomial ring, and let G act so as to preserve degrees. Then $R=S^G$ will be generated over K by finitely many forms of S, and we can write $R \simeq K[z_0, \cdots, z_t]/I$. Here, we assume that $z_0, \cdots, z_t$ map to generating forms of R, and we grade $T=K[z_0, \cdots, z_t]$ so that the K-homomorphism preserves degrees. Then I will be a homogeneous ideal of T, and is the solution to the "second main problem of invariant theory" (cf. [**14**, Chapter II, C]) for this particular representation. In this situation the assertion that R ($\simeq T/I$) is Cohen-Macaulay is equivalent to the assertion that I is *perfect*, i.e. $\mathrm{pd}_T\, T/I=\text{grade } I$. Let $\mathscr{K}$ be a graded T-free minimal resolution of T/I. Then from the Main Theorem and, for example, Theorem 3 and Corollary 1.2 of [**4**], we have

COROLLARY 1. *With notation as above, so that $S^G \simeq T/I$, the length of $\mathscr{K}$ is $g=$grade $I=$height I, and $\mathscr{K}$ is grade-sensitive. That is, if $u_0, \cdots, u_t$ are elements of a Noetherian K-algebra B, and we make B into a T-algebra by means of the homomorphism h which takes z_i to u_i, $0 \leqq i \leqq t$, then if $J=h(I)B$ and E is any B-module of finite type such that $JE \neq E$, then the grade of J on E is the number of vanishing homology groups, counting from the left, of the complex $\mathscr{K} \otimes_T E$. In particular, if the grade of J on E is equal to g, then $\mathscr{K} \otimes_T E$ is acyclic.*

COROLLARY 2. *With notation as in Corollary* 1, *let $E=B$. Then every minimal prime of $J=h(I)B$ has height at most g, and if the grade of J is as large as possible, i.e. g, then J is perfect ($\mathscr{K} \otimes_T B$ is acyclic and gives a resolution of length g) and hence all the associated primes of J have grade g. If J has grade g and B is Cohen-Macaulay, then the associated primes of J all have height g and B/J is again Cohen-Macaulay.*

We also note

COROLLARY 3. *If K has characteristic* 0 *and G is semisimple and acts on $S=K[y_0, \cdots, y_m]$ so as to preserve the grading, then $R=S^G$ is a Cohen-Macaulay UFD and, hence, Gorenstein.*

3. **Remarks on the proof of the Main Theorem.** When G is linearly reductive there is an R-module retraction (the Reynolds operator) of S onto $R=S^G$. This fact is crucial in the proof of the Main Theorem. The proof goes roughly like this: First, we reduce to the case where G is connected and then the theorem is stated slightly more generally—S is only assumed regular at G-invariant primes. Utilizing devices involving associated graded rings and generalized Rees rings we make a reduction to a sort of "minimal" graded case: S is the symmetric algebra of a projective module E over a domain B (where G acts on B, E and B is a K-algebra) and B has no G-invariant ideals except 0, B. $R=S^G$ is a finitely generated graded algebra over a field, B^G, and $R_{\mathscr{P}}$ is Cohen-Macaulay except possibly when $\mathscr{P}=P$, the irrelevant maximal ideal (this last condition is what we meant by "minimal"). Let L be the field of fractions of B. Then $L \otimes_B S=L[y_0, \cdots, y_m]$ is a polynomial ring. R is a direct summand of S (via the Reynolds operator) and this turns out to imply that R is pure in $L \otimes_B S$. Because of this purity, the maps described in Proposition A, which are zero by Proposition A, are also injective, and one finds that $H^i(X, \mathscr{O}_X)=0$, $i \geqq 1$, where $X=\operatorname{Proj}(R)$. [We note that Proposition A itself is proved by a reduction to characteristic p.] By "minimality" the local rings of X are Cohen-Macaulay and one can use Serre-Grothendieck duality to show that $R^{(d)}=\sum \bigoplus_{n \geqq 0} R_{nd}$ is Cohen-Macaulay for all large d. In the final stages of the proof, we show that R itself is Cohen-Macaulay by reducing to characteristic p a second time. (An important point is that in characteristic p we can take $d=p^e$ for large e and then embed $R \to R^{(d)}$ by using the Frobenius.) A key technical lemma which we use repeatedly in the reductions to characteristic p and which generalizes the usual statements about generic flatness is

LEMMA. *Let A be a Noetherian domain, R an A-algebra of finite type, S an R-algebra of finite type, E an S-module of finite type, and M an R-submodule of E of finite type. Then there is an $a \in A-\{0\}$ such that E_a/M_a is A_a-free.*

The proof of the characteristic p Theorem is easier and uses local cohomology analogues of the arguments in the proof of the Main Theorem.

4. **Concluding remarks.**

REMARK 1. The regularity of S is essential in the statement of the Main Theorem. There are counterexamples when $G=\mathrm{GL}(1, K)$ and S is a graded Cohen-Macaulay Gorenstein UFD. But the regularity of S is used in an apparently rather nongeometric way: it is only used to show that the Frobenius in a certain auxiliary ring, after reducing to characteristic $p>0$, is flat (cf. [**8**]).

REMARK 2. The fact that reduction to characteristic p seems to be essential in the proof of the Main Theorem is odd, because the Main Theorem is primarily a characteristic 0 theorem. There are very few linearly reductive groups in characteristic $p>0$. See **[12]**. We note that in **[13]** techniques related to ours are used to settle a number of other questions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address (Melvin Hochster): Mathematisch Institut, Aarhus University, Aarhus, Denmark

(After July 1, 1974) Department of Mathematics, Purdue University, West LaFayette, Indiana 47907