

FIXED POINTS OF ENDOMORPHISMS OF COMPACT GROUPS

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1. Introduction. Let G be a compact, connected Lie group and denote its real Čech cohomology by $H^*(G)$. Then $H^*(G)$ is an exterior algebra with generators $1=z_0, z_1, z_2, \dots, z_\lambda$; where, by a theorem of Hopf [3], λ is equal to the rank of G (the dimension of a maximal torus). This paper announces some improvements of Hopf's result. The details will be published elsewhere.

2. Fixed point groups. For a set X and a function $f: X \rightarrow X$, let $\Phi(f)$ denote the set of fixed points of f : those $x \in X$ for which $f(x)=x$. If X is a topological group and f is a homomorphism, we will use the symbol $\Phi_0(f)$ for the component of the group $\Phi(f)$ which contains the identity element of X .

We consider a compact, connected Lie group G and let h be an automorphism of G . Choose algebra generators $1=z_0, z_1, z_2, \dots, z_\lambda$ for $H^*(G)$ and let $\mathbf{H}^*(G)$ denote the linear span of $z_1, z_2, \dots, z_\lambda$. The automorphism h^* of $H^*(G)$ induced by h takes $\mathbf{H}^*(G)$ to itself; let \mathbf{h}^* denote the restriction of h^* to $\mathbf{H}^*(G)$.

Our main result is

THEOREM 1. *Let G be a compact, connected Lie group and let h be an automorphism of G . Then the rank of the Lie group $\Phi_0(h)$ is equal to the dimension of the vector space $\Phi(\mathbf{h}^*)$.*

Note that Theorem 1 reduces to Hopf's theorem when h is the identity function.

One might suspect that Theorem 1 is a consequence of some more intimate relationship between $H^*(\Phi_c(h))$ and $\Phi(h^*)$. However, let $g \in G$ be a regular element and define $h(x)=g^{-1}xg$, for $x \in G$, then h induces the identity isomorphism in cohomology, so $\Phi(h^*)=H^*(G)$; while $\Phi_0(h)$ is a maximal torus of G . Thus the possibilities for such a relationship are very limited.

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The proof of Theorem 1 can be reduced to two special cases: when G is abelian and when G is semisimple.

In the abelian case we can work in a more general setting. Let G be a compact, connected abelian topological group and let h be an endomorphism of G . Denote the character group of G by \hat{G} and write the endomorphism of \hat{G} induced by h as h^\wedge . Using the techniques of Pontryagin duality theory, we prove

THEOREM 2. *Let h be an endomorphism of a compact, connected abelian topological group G , then the dimension of the group $\Phi_0(h)$ is equal to the torsion-free rank of $\Phi(h^\wedge)$.*

Let $h^{* \cdot 1}$ be the endomorphism of $H^1(G)$ induced by h . Representation theory and the continuity of Čech cohomology are used to prove

THEOREM 3. *Let h be an endomorphism of a compact, connected abelian topological group G , then the torsion-free rank of $\Phi(h^\wedge)$ is equal to the dimension of the vector space $\Phi(h^{* \cdot 1})$.*

Theorems 2 and 3 imply Theorem 1 in the case that G is abelian.

If h is an automorphism of a compact, connected semisimple Lie group G , then h^m (h composed with itself m times) is an inner automorphism, for some $m \geq 1$. We may assume that h^m is, in fact, the identity automorphism. Let Γ be the semidirect product, of G and the cyclic group of order m , induced by h . Then there is an element $\gamma \in \Gamma$ such that $h(x) = \gamma^{-1}x\gamma$ for all $x \in \Gamma_0 = G$. The proof of Theorem 1, in the case that G is semisimple, now follows from Theorem 4.3 of [1].

3. A bound on the rank. Let \mathfrak{A} be a simple Lie algebra and define $\rho(\mathfrak{A})$ to be the minimum rank of $\Phi(\eta)$, for all automorphisms η of \mathfrak{A} . The numbers $\rho(\mathfrak{A})$ are easily computed using Theorem 1 and material from [2].

THEOREM 4. *Let G be a compact, connected Lie group with Lie algebra \mathfrak{G} . Write*

$$\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A}_1^1 \oplus \cdots \oplus \mathfrak{A}_1^{k(1)} \oplus \cdots \oplus \mathfrak{A}_u^1 \oplus \cdots \oplus \mathfrak{A}_u^{k(u)}$$

where \mathfrak{Z} is abelian, $\mathfrak{A}_s^i \cong \mathfrak{A}_s^j \cong \mathfrak{A}_s$ for each $s=1, 2, \dots, u$ and all $i, j=1, \dots, k(s)$, and $\mathfrak{A}_s^i \not\cong \mathfrak{A}_t^j$ if $s \neq t$. Then

$$\sum_{s=1}^u \rho(\mathfrak{A}_s) \leq \text{rank } \Phi_0(h)$$

for all automorphisms h of G .

If, for example, G is simply-connected, then there is an automorphism h of G such that $\text{rank } \Phi_0(h)$ is precisely $\sum_{s=1}^u \rho(\mathfrak{A}_s)$; so Theorem 4 cannot be improved in general.

COROLLARY 4.1 (DE SIEBENTHAL [4]). *If G is a compact, connected Lie group and there is an automorphism of G with a finite set of fixed points, then G is abelian.*

COROLLARY 4.2. *If there is an automorphism h of a compact, connected Lie group G such that $\Phi_0(h)$ is a sphere, then either G is abelian or its Lie algebra \mathfrak{G} is of the form $\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$ where \mathfrak{Z} is abelian and \mathfrak{A} is a simple Lie algebra, either of type A_1 or of type A_2 .*

4. The power map. Let G be a Lie group whose components are compact. In other words, G is an extension of a compact, connected Lie group G_0 by a discrete, but not necessarily finite, group. Define the *rank* of a component C of G to be the rank of the identity component of the centralizer of g in G , for any $g \in C$. Theorem 1 implies that the definition is independent of the choice of $g \in C$.

Define, for $k \geq 2$, the "power map" $p_k: G \rightarrow G$ by $p_k(g) = g^k$. The component of G containing an element g is gG_0 , so $p_k(gG_0) \subseteq g^k G_0$.

THEOREM 5. *Let G be a Lie group with compact components. The following are equivalent:*

- (i) $p_k(gG_0) = g^k G_0$,
- (ii) the degree of the map $p_k: gG_0 \rightarrow g^k G_0$ is not zero,
- (iii) $\text{rank}(gG_0) = \text{rank}(g^k G_0)$.

Theorem 5 extends the main result, Theorem 5.2, of [1]—for compact Lie groups—to Lie groups with compact components since, when G is compact, the definition of the rank of a component given above agrees with the definition used in [1].

The equivalence of (ii) and (iii) follows easily from Theorem 1 above and Theorem 2.3 of [1]. Of course (ii) implies (i). The rest of the proof—if the degree of $p_k: gG_0 \rightarrow g^k G_0$ is zero then the dimension of $p_k(gG_0)$ is smaller than the dimension of G —reduces to the usual cases: G_0 abelian and G_0 semisimple. Following a suggestion of K. H. Hofmann, we consider the map $\varphi_k^g: G_0 \rightarrow G_0$ defined by $\varphi_k^g(x) = g^{-k}(gx)^k$ and prove the equivalent statement: if the degree of φ_k^g is zero, then $\varphi_k^g(G_0)$ has smaller dimension than G_0 . In case G_0 is abelian, we again use Pontryagin duality theory. When G_0 is semisimple we can assume that g^m is in the centralizer of G_0 , for some $m \geq 1$. This permits us to construct a Lie group with identity component G_0 and only m components, apply Theorem 5.2 of [1] to

that compact group, and then “lift” that information back to G to obtain the desired result.

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