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HOLOMORPHY OF COMPOSITION

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1. **Introduction.** We wish to consider the following two problems for E, F, G Banach spaces over the complex field $\boldsymbol{C}$ and $\mathscr{H}(E; F)$, $\mathscr{H}(F; G)$, $\mathscr{H}(E; G)$ the corresponding spaces of holomorphic functions between them (we follow the definitions and notation given in [**3**]): (1) For what vector subspaces $X \subset \mathscr{H}(E; F)$, $Y \subset \mathscr{H}(F; G)$, $Z \subset \mathscr{H}(E; G)$ and corresponding locally convex topologies τ_X, τ_Y, τ_Z will the composition $\phi: (f, g) \in (X, \tau_X) \times (Y, \tau_Y) \to g \circ f \in (Z, \tau_Z)$ be holomorphic? (2) Investigate the holomorphy of $\phi: \mathscr{H}(U; V) \times \mathscr{H}(V; W) \to \mathscr{H}(U; W)$ for $U \subset E$, $V \subset F$, $W \subset G$ open. We are driven to consider general locally convex topologies on X, Y, Z since if ϕ holomorphic means it is separately continuous, then, in particular, the evaluation $f \in (\mathscr{H}(F; \boldsymbol{C}), \tau) \mapsto f(x) \in \boldsymbol{C}$ is continuous. But from [**1**] and [**2**], if F is, for example, a separable or reflexive infinite dimensional Banach space, then τ is not first countable.

2. **Definitions of holomorphy** [**4**]. Let X and Y be complex locally convex spaces (LCS), and W an open, nonempty subset of X. Then $f: W \to Y$ is said to be *holomorphic* if for every $\xi \in W$ there is a sequence $P_m \in \mathscr{P}(^m X; Y)$ (the space of continuous m-homogeneous polynomials from X to Y), $m=0, 1, \cdots$, such that for each continuous seminorm β on Y, one can find a neighborhood V of ξ in W for which

$$\lim_{M \to \infty} \beta\left[f(x) - \sum_{m=0}^{M} P_m(x - \xi) \right] = 0$$

uniformly for $x \in V$. f is said to be *G-holomorphic* (provided X is Hausdorff) if for each $\xi \in W$, $x \in X$, the map $\lambda \in V \mapsto f(\xi + \lambda x) \in Y$ is holomorphic, where $V = \{\lambda \in \boldsymbol{C}: \xi + \lambda x \in W\}$. We denote the space of holomorphic (G-holomorphic) maps by $\mathscr{H}(W; Y)$ ($\mathscr{H}_G(W; Y)$). f is said to be *amply bounded* if for each continuous seminorm β on Y, $\beta \circ f$ is locally bounded.

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If f is continuous, or locally bounded, then it is amply bounded. The space of amply bounded maps is denoted $\mathscr{A}\mathscr{B}(W; Y)$. Then

$$\mathscr{H}_G(W; Y) \cap \mathscr{A}\mathscr{B}(W; Y) = \mathscr{H}(W; Y).$$

3. **Topologies.** We shall consider the locally convex topologies on $\mathscr{H}(U; F)$ (E, F Banach spaces, $U \subset E$ open, nonempty) as given in [**3**]. In particular, τ_0 denotes the compact-open topology, τ_ω the topology of seminorms ported by compact subsets of U, τ_λ the topology of seminorms ported by all open covers of U, and τ_δ the bornological topology associated with τ_0. We let $\mathscr{H}_b(U; F)$ be the space of holomorphic functions of bounded type with its natural topology τ_{0b}.

We have the following chain of inequalities $\tau_0 \leqq \tau_\infty \leqq \tau_\sigma \leqq \tau_\pi \leqq \tau_\omega \leqq \tau_\delta$ and $\tau_\delta | \mathscr{H}_b \leqq \tau_{0b}$. $\tau_\delta | \mathscr{H}_b = \tau_{0b}$, that is, τ_{0b} is the bornological topology associated with $\tau_0 | \mathscr{H}_b$ if and only if (for U ξ-balanced) $\mathscr{H}_b(U; F) = \mathscr{H}(U; F)$. Dineen [**2**] has shown, however, that if in the dual of E every bounded sequence has a weak* convergent subsequence, for example if E is separable or reflexive, then $\mathscr{H}_b(E; \mathbf{C}) \neq \mathscr{H}(E; \mathbf{C})$, and so $\mathscr{H}_b(U; F) \neq \mathscr{H}(U; F)$.

4. **Basic setting for the problem.** We consider first Problem 2. Assume $U \subset E$, $V \subsetneqq F$ are open and nonempty. To avoid manifolds we need $\mathscr{H}(U; V)$ open in $\mathscr{H}(U; F)$ or a vector subspace, but the latter occurs exactly when $V = F$.

PROPOSITION 1. *If $U = E$, or if $\mathscr{H}_b(E; \mathbf{C}) \neq \mathscr{H}(E; \mathbf{C})$ when $U \neq E$, then $\mathscr{H}(U; V)$ is not open in $(\mathscr{H}(U; F), \tau_\lambda)$.*

For $A \subset U$ and $\mathscr{F} \subset \mathscr{H}(U) = \mathscr{H}(U; \mathbf{C})$, we define the $\mathscr{F}$-convex hull of A to be

$$\hat{A}_{\mathscr{F}} = \{x \in U : |f(x)| \leqq |f|_A \quad \text{for all } f \in \mathscr{F}\},$$

where $|f|_A = \sup\{|f(x)| : x \in A\}$. U is said to be $\mathscr{H}(U)$-convex (resp. $\mathscr{H}_b(U)$-convex) if for every compact (resp. U-bounded) subset K of U, $\hat{K}_{\mathscr{H}(U)}$ (resp. $\hat{K}_{\mathscr{H}_b(U)}$) is compact (resp. U-bounded), where A is a U-bounded subset of U if it is bounded (in E) and, if $U \neq E$, the distance from A to the boundary of U is not zero. If U is convex (in particular, all of E), then it is $\mathscr{H}_b(U)$-convex and so $\mathscr{H}(U)$-convex.

PROPOSITION 2. *If U is $\mathscr{H}_b(U)$-convex, then $\mathscr{H}_b(U; V)$ is not open in $(\mathscr{H}_b(U; F), \tau_{0b})$.*

PROPOSITION 3. *If U is $\mathscr{H}(U)$-convex, then $\mathscr{H}(U; V)$ is not open in $(\mathscr{H}(U; F), \tau_\omega)$.*

Hence, the setting for the problem we shall choose is to consider $X \subset \mathscr{H}(U; F)$, $Y \subset \mathscr{H}(F; G)$, and $Z \subset \mathscr{H}(U; G)$.

5. G-**holomorphy of** ϕ. We investigate the holomorphy of ϕ by examining separately when it is G-holomorphic and amply bounded. We may reduce the problem by using a theorem of Nachbin **[4]** which implies that if M is a LCS, W an open subset of M, and $\tau_1(N) \leqq \tau_2(N)$ locally convex topologies on a vector space N such that the $\tau_1(N)$-closure of every $\tau_2(N)$-bounded set is $\tau_2(N)$-bounded (designated condition (A)), then

$$\mathscr{H}_G(W; N_1) \cap \mathscr{A}\mathscr{B}(W; N_2) = \mathscr{H}(W; N_2)$$

where $N_i = (N, \tau_i(N))$ for $i=1, 2$. Condition (A) is implied by (B): every $\tau_1(N)$-bounded subset of N is $\tau_2(N)$-bounded, or (C): $\tau_2(N)$ is locally $\tau_1(N)$-closed (that is, $\tau_2(N)$ has a base of neighborhoods of zero which are $\tau_1(N)$-closed).

Set $W = (X, \tau_X) \times (Y, \tau_Y)$ where $X \subset \mathscr{H}(U; F)$, $Y \subset \mathscr{H}(F; G)$ are vector subspaces, and $N_1 = (\mathscr{H}(U; G), \tau_0)$. Since $\tau_{0b}(\mathscr{H}_b(U; G))$ is locally $\tau_0(\mathscr{H}_b(U; G))$-closed, so (C) applies, and $\tau_\delta(\mathscr{H}(U; G))$ is the bornological topology associated with $\tau_0(\mathscr{H}(U; G))$, so (B) applies, and since all the topologies introduced above lie between τ_{0b} or τ_δ and τ_0, then it suffices only to show ϕ is amply bounded for the given topologies, since it is G-holomorphic for all locally convex Hausdorff topologies τ_X, τ_Y when $\tau_1(N) = \tau_0$.

6. **Ample boundedness of** ϕ. Let $\mathscr{M}$ be a collection of subsets of U. Let $X_{\mathscr{M}}$ be the space of holomorphic functions in $X \subset \mathscr{H}(U; F)$ which are bounded on each $W \in \mathscr{M}$, and give it the LCS topology defined by the family of seminorms $(|\cdot|_W)_{W \in \mathscr{M}}$. Let $Z_{\mathscr{M}}$ be defined similarly for $Z \subset \mathscr{H}(U; G)$, and let Y be a vector subspace of $\mathscr{H}(F; G)$. Let J_ε designate a collection of subsets of F of the form $J_\varepsilon(X_{\mathscr{M}}, \mathscr{M}) = \{B_{\varepsilon(f,W)}(f(W)) : f \in X_{\mathscr{M}}, W \in \mathscr{M}\}$ where $\varepsilon : X_{\mathscr{M}} \times \mathscr{M} \to \boldsymbol{R}^+$ and $B_r(A) = A + r\{x : \|x\| < 1\}$. Then the basic result is

PROPOSITION 4. *If X contains all the constant functions, then ϕ: $X_{\mathscr{M}} \times (Y, \tau_Y) \to Z_{\mathscr{M}}$ is amply bounded if and only if there is an open cover J_ε of F such that $(Y, \tau_Y) \subset \mathscr{H}(F; G)_{J_\varepsilon}$ continuously. This last implies $\tau \geqq \tau_\lambda(Y)$.*

PROPOSITION 5. (i) *If $\phi : \mathscr{H}(U; F)_{\mathscr{M}} \times (\mathscr{H}(F; G), \tau_Y) \to \mathscr{H}(U; G)_{\mathscr{M}}$ is amply bounded, then $\mathscr{H}(F; G) = \mathscr{H}_b(F; G)$ (and $\tau_Y \geqq \tau_\lambda$).*

(ii) *If $\tau_Y \geqq \tau_{0b}$, then the converse of* (i) *is true.*

For example, taking $\mathscr{M}$ in Proposition 4 to be the compact (resp. U-bounded) subsets of U yields τ_0 (resp. τ_{0b}). Arguing directly, we also get $\phi : (\mathscr{H}(U; F), \tau) \times (\mathscr{H}_b(F; G), \tau_{0b}) \to (\mathscr{H}(U; G), \tau)$ is amply bounded when $\tau = \tau_\infty$, τ_σ, and (when U is ξ-balanced) τ_ω.

REMARKS. We may repeat the above investigation for E, F, G locally convex spaces instead of just Banach spaces. If F is Hausdorff and G seminormed, then the generalized form of Proposition 5 yields ϕ: $(\mathscr{H}(U; F), \tau_0) \times (\mathscr{H}(F; G), \tau_Y) \to (\mathscr{H}(U; G), \tau_0)$ amply bounded implies F is normable and $\mathscr{H}(F; G) = \mathscr{H}_b(F; G)$.

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