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FOLIATIONS AND GROUPS OF DIFFEOMORPHISMS

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John Mather has described a close relation between framed codimension-one Haefliger structures (these form a class of singular foliations), and the group of compactly supported diffeomorphisms of $\boldsymbol{R}^1$, with discrete topology [**11**], [**12**], [**14**]. In this announcement I will describe generalizations of his ideas to higher codimension Haefliger structures and groups of diffeomorphisms of arbitrary manifolds. See Haefliger [**7**] for a development of Haefliger structures and their classifying spaces.

I would like to thank Boyd Anderson, André Haefliger and John Mather for long discussions and proddings about this material, and many others for helpful conversations and relevant information.

Let $\mathrm{Diff}^r(M^p)$ denote the group of C^r diffeomorphisms of M^p, a closed manifold. Let $\mathrm{Diff}_0^r(M^p)$ denote the connected component of the identity.

THEOREM 1. $\mathrm{Diff}_0^\infty(M^p)$ *is a simple group.*

The proof makes use of both the theorem of Epstein [**4**] that the commutator subgroup of $\mathrm{Diff}_0(M^p)$ is simple, and of the result of M. Herman [**9**] which gives the case M^p is a p-torus.

THEOREM 2. $B\bar{\Gamma}_p^\infty$ *is* $(p+1)$*-connected, where* $B\bar{\Gamma}_p^\infty$ *is the classifying space for framed, codimension* p, C^∞, *Haefliger structures.*

The more usual notation is $F\Gamma_p^\infty = B\bar{\Gamma}_p^\infty$. Haefliger proved [**6**] that $B\bar{\Gamma}_p^r$ is p-connected for $1 \leqq r \leqq \infty$; Mather proved that $B\bar{\Gamma}_1^\infty$ is 2-connected.

Theorem 2 means that two C^∞ foliations of a manifold coming from nonsingular vector fields are homotopic as Haefliger structures if and only if the normal bundles are isomorphic.

Theorems 1 and 2 are proven by showing they are related; cf. Theorem 4 for a statement of a relationship.

COROLLARY. $P_1^{[p/2]}$ *is nontrivial in* $H^*(B\Gamma_p^\infty; \boldsymbol{R})$ *where* P_1 *is the first real Pontrjagin class of the normal bundle to the canonical Haefliger structure.*

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Thus, Bott's vanishing theorem [**1**], which says real Pontrjagin classes in $B\Gamma_p^r$ $(r \geqq 2)$ vanish above dimension $2p$, gives a sharp bound on dimensions.

This corollary in the case $p=2$ follows easily from Theorem 2.

For higher codimensions, product foliations then yield examples.

THEOREM 3. *$B\bar{\Gamma}_p^0$ is contractible.*

Again, Mather proved this when the codimension is one.

This means topological Haefliger structures are completely determined up to homotopy by their normal micro-bundles.

Theorem 3 implies that Bott's vanishing theorem is quite false in the topological case—any normal micro-bundle is the normal micro-bundle for a topological foliation. In fact, if the micro-bundle is differentiable, it even admits a Haefliger structure Lipschitz close to being differentiable.

A little background and notation is necessary before the statement of the more general relationships. Let G be a topological group. Let G_δ be G with discrete topology. Then the map $G_\delta \to G$ is a continuous map which has a homotopy-theoretic fiber $\bar{G}$. $\bar{G}$ is also a topological group: the explicit construction for $\bar{G}$ is the space of paths α in G ending at the identity $e=\alpha(1)$, with discrete topology on $\alpha(0)$. Then multiplication is pointwise. There are maps, now,

$$\bar{G} \to G_\delta \to G \to B\bar{G} \to BG_\delta \to BG,$$

and any two consecutive arrows define a fibration.

BG is the classifying space for G-bundles. BG_δ classifies flat G-bundles: for instance, $B\,\mathrm{Diff}^\infty(M^n)_\delta$ has an associated M-bundle, with discrete structure group: i.e., a C^∞ foliation transverse to the fibers of the bundle. Thus, $B\,\mathrm{Diff}^\infty(M^p)_\delta$ classifies "foliated M^p-bundles". Finally, $B\bar{G}$ classifies G-bundles with a flat structure, together with a global trivialization defined (up to homotopy); e.g. $B\,\overline{\mathrm{Diff}}^\infty(M^p)$ classifies "foliated M^p-products".

Let $\mathrm{Diff}_K(\boldsymbol{R}^p)$ be the group of diffeomorphisms of $\boldsymbol{R}^p$ with compact support. Then again, $B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\times\boldsymbol{R}^p$ has a foliation of codimension p transverse to the $\boldsymbol{R}^p$-factors. Thus, there is a classifying map

$$B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p) \times \boldsymbol{R}^p \to B\bar{\Gamma}_p^r.$$

(The image is in $B\bar{\Gamma}_p^r$ since there is a natural trivialization of the normal bundle to the foliation.)

The foliation agrees with the trivial, product foliation in a neighborhood of ∞ in the $\boldsymbol{R}^p$ factors. Thus, one obtains a map of the p-fold suspension of $B\,\overline{\mathrm{Diff}}_K^r\boldsymbol{R}^p$,

$$S^p(B\,\overline{\mathrm{Diff}}_K^r\boldsymbol{R}^p) \to B\bar{\Gamma}_p^r.$$

This defines an adjoint map $B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}^r_p)$ to the p-fold loop space of $B\bar{\Gamma}^r_p$.

THEOREM 4. *The map* $B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}^r_p)$ *induces an isomorphism on homology.*

This theorem is due to Mather in the case $p=1$.

The map is certainly not a homotopy equivalence since $\pi_1(B\,\overline{\mathrm{Diff}}{}^r_K\boldsymbol{R}^p)$ is highly nonabelian while $\pi_1(\Omega^pB\bar{\Gamma}^r_p)=\pi_{p+1}(B\bar{\Gamma}^r_p)$ is abelian.

Similarly, there is a map $B\,\overline{\mathrm{Diff}}{}^r(M^p)\times M^p\to B\Gamma^r_p$ which is a lifting of the classifying map for the tangent bundle of M^p, so there is a commutative diagram

$$\begin{array}{ccc} & & B\Gamma^r_p \\ & \nearrow & \downarrow \\ B\,\overline{\mathrm{Diff}}{}^r(M^p)\times M^p & \to & BO_p \end{array}$$

Let X be the space of liftings of the classifying map for $T(M^p)$ in BO_p to $B\Gamma^r_p$. Then we have a map $B\,\overline{\mathrm{Diff}}{}^r(M^p)\to X$.

THEOREM 5. *The map*

$$B\,\overline{\mathrm{Diff}}{}^r(M^p)\to X$$

induces an isomorphism on homology.

Again, this is not a homotopy equivalence since $\pi_1(X)$ is abelian.

For the case $r=0$, we assume M^p is a differentiable manifold.

COROLLARY. (a) $B\,\overline{\mathrm{Homeo}}(M^p)$ *is acyclic, where* $\mathrm{Homeo}(M^p)=\mathrm{Diff}^0(M^p)$ *is the group of homeomorphisms of* M^p.

(b) *The map* $B\,\mathrm{Homeo}(M^p)_\delta\to B\,\mathrm{Homeo}(M^p)$ *induces an isomorphism on homology.*

This corollary is implied by Theorems 3 and 5. Cf. Mather **[13]**, who showed $B\,\mathrm{Homeo}_K(\boldsymbol{R}^p)_\delta$ is acyclic.

COROLLARY. *The following groups are isomorphic, where k is the first positive integer such that one of them is nontrivial*:

(i) $H_k(B\,\overline{\mathrm{Diff}}{}^r(M^p);Z)$,

(ii) $H_k(B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p);Z)$,

(iii) $H_{k+p}(B\bar{\Gamma}^r_p;Z)$.

CONJECTURE. *This first k is* $p+1$, *for* $r=\infty$.

Mather's theorem **[11]** shows this for $p=1$. Bott and Haefliger showed that all differentiable characteristic classes (in some sense) vanish below this dimension, $H_{2p+1}(B\bar{\Gamma}_p^r; Z)$ [2], [3].

In [16] I sketched examples showing there is a surjective homomorphism

$$H_3(B\bar{\Gamma}_1^\infty; Z) \twoheadrightarrow \boldsymbol{R},$$

using the Godbillon-Vey invariant gv [5]. Recently I have extended this to arbitrary codimension, so there is a surjective homomorphism

$$H_{2p+1}(B\bar{\Gamma}_p^\infty; Z) \twoheadrightarrow \boldsymbol{R}.$$

Bibliography

1. R. Bott, *On a topological obstruction to integrability*, Proc. Int. Congress Nice, 1970, 27–36.

2. R. Bott and A. Haefliger, *On characteristic classes of* Γ*-foliations*, Bull. Amer. Math. Soc. **78** (1972), 1039–1044.

3. ———, *Continuous cohomology and characteristic classes*, (to appear).

4. D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. **22** (1970), 165–173. MR **42** #2491.

5. C. Godbillon and J. Vey, *Un invariant des feuilletage de codimension* 1, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A92–A95. MR **44** #1046.

6. A. Haefliger, *Feuilletages sur les variétés ouvertes*, Topology **9** (1970), 183–194. MR **41** #7709.

7. ———, *Homotopy and integrability*, Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Math., vol. 197, Springer, Berlin, 1971, pp. 133–163. MR **44** #2251.

8. ———, *Sur les classes characteristiques des feuilletages*, Séminaire Bourbaki, No. 412, June 1972.

9. M. Herman, *Simplicité du groupe des difféomorphismes de classe* C^∞, *isotopes à l'identité, du tore de dimension n*, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A232–A234. MR **44** #4788.

10. M. Herman and F. Sergeraert, *Sur un théorème d'Arnold et Kolmogorov*, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A409–A411. MR **44** #7586.

11. J. Mather, *On Haefliger's classifying space*. I, Bull. Amer. Math. Soc. **77** (1971), 1111–1115. MR **44** #1047.

12. ———, *On Haefliger's classifying space*. II: *Approximation theorems*, (preprint).

13. ———, *The vanishing of homology of certain groups of homeomorphisms*, Topology **10** (1971), 297–298. MR **44** #5973.

14. ———, *Integrability in codimension* 1, Comment. Math. Helv. (to appear).

15. W. Thurston, *Noncobordant foliations of* S^3, Bull. Amer. Math. Soc. **78** (1972), 511–514. MR **45** #7741.

16. ———, *Variation of the Godbillon-Vey invariant in higher codimension*, (to appear).

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