FOLIATIONS AND GROUPS OF DIFFEOMORPHISMS

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John Mather has described a close relation between framed codimension-one Haefliger structures (these form a class of singular foliations), and the group of compactly supported diffeomorphisms of $\mathbb{R}^1$, with discrete topology [11], [12], [14]. In this announcement I will describe generalizations of his ideas to higher codimension Haefliger structures and groups of diffeomorphisms of arbitrary manifolds. See Haefliger [7] for a development of Haefliger structures and their classifying spaces.

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Let $\text{Diff}^r(M^p)$ denote the group of $C^r$ diffeomorphisms of $M^p$, a closed manifold. Let $\text{Diff}_0^r(M^p)$ denote the connected component of the identity.

**Theorem 1.** $\text{Diff}_0^r(M^p)$ is a simple group.

The proof makes use of both the theorem of Epstein [4] that the commutator subgroup of $\text{Diff}_0^r(M^p)$ is simple, and of the result of M. Herman [9] which gives the case $M^p$ is a $p$-torus.

**Theorem 2.** $B\Gamma_p^\infty$ is $(p+1)$-connected, where $B\Gamma_p^\infty$ is the classifying space for framed, codimension $p$, $C^\infty$, Haefliger structures.

The more usual notation is $F\Gamma_p^\infty=B\Gamma_p^\infty$. Haefliger proved [6] that $B\Gamma_p^r$ is $p$-connected for $1 \leq r \leq \infty$; Mather proved that $B\Gamma_1^\infty$ is 2-connected.

Theorem 2 means that two $C^\infty$ foliations of a manifold coming from nonsingular vector fields are homotopic as Haefliger structures if and only if the normal bundles are isomorphic.

Theorems 1 and 2 are proven by showing they are related; cf. Theorem 4 for a statement of a relationship.

**Corollary.** $P_1^{[p/2]}$ is nontrivial in $H^*(B\Gamma_p^\infty; \mathbb{R})$ where $P_1$ is the first real Pontrjagin class of the normal bundle to the canonical Haefliger structure.
Thus, Bott’s vanishing theorem \[1\], which says real Pontrjagin classes in \(B^{r-2}P_r\) vanish above dimension \(2p\), gives a sharp bound on dimensions.

This corollary in the case \(p=2\) follows easily from Theorem 2.

For higher codimensions, product foliations then yield examples.

**THEOREM 3.** \(B\Gamma_p\) is contractible.

Again, Mather proved this when the codimension is one.

This means topological Haefliger structures are completely determined up to homotopy by their normal micro-bundles.

Theorem 3 implies that Bott’s vanishing theorem is quite false in the topological case—any normal micro-bundle is the normal micro-bundle for a topological foliation. In fact, if the micro-bundle is differentiable, it even admits a Haefliger structure Lipschitz close to being differentiable.

A little background and notation is necessary before the statement of the more general relationships. Let \(G\) be a topological group. Let \(G_δ\) be \(G\) with discrete topology. Then the map \(G_δ \to G\) is a continuous map which has a homotopy-theoretic fiber \(\tilde{G}\). \(\tilde{G}\) is also a topological group: the explicit construction for \(\tilde{G}\) is the space of paths \(x\) in \(G\) ending at the identity \(e=\alpha(1)\), with discrete topology on \(\alpha(0)\). Then multiplication is pointwise. There are maps, now,

\[
\tilde{G} \to G_δ \to G \to B\tilde{G} \to B\tilde{G}_δ \to BG,
\]

and any two consecutive arrows define a fibration.

\(BG\) is the classifying space for \(G\)-bundles. \(BG_δ\) classifies flat \(G\)-bundles: for instance, \(B\text{Diff}^\infty(M^n)\) has an associated \(M\)-bundle, with discrete structure group: i.e., a \(C^0\) foliation transverse to the fibers of the bundle. Thus, \(B\text{Diff}^\infty(M^n)_δ\) classifies “foliated \(M^n\)-bundles”. Finally, \(BG\) classifies \(G\)-bundles with a flat structure, together with a global trivialization defined (up to homotopy); e.g. \(B\text{Diff}^\infty(M^n)\) classifies “foliated \(M^n\)-products”.

Let \(\text{Diff}_K(R^p)\) be the group of diffeomorphisms of \(R^p\) with compact support. Then again, \(B\text{Diff}_K^\infty(R^p) \times R^p\) has a foliation of codimension \(p\) transverse to the \(R^p\)-factors. Thus, there is a classifying map

\[
B\text{Diff}_K^\infty(R^p) \times R^p \to B\Gamma^p_p.
\]

(The image is in \(B\Gamma^p_p\) since there is a natural trivialization of the normal bundle to the foliation.)

The foliation agrees with the trivial, product foliation in a neighborhood of \(\infty\) in the \(R^p\) factors. Thus, one obtains a map of the \(p\)-fold suspension of \(B\text{Diff}_K^\infty R^p\),

\[
S^p(B\text{Diff}_K^\infty R^p) \to B\Gamma^p_p.
\]
This defines an adjoint map $B \text{Diff}^r_K(R^p) \to \Omega^p(B\Gamma^r_p)$ to the $p$-fold loop space of $B\Gamma^r_p$.

**Theorem 4.** The map $B \text{Diff}^r_K(R^p) \to \Omega^p(B\Gamma^r_p)$ induces an isomorphism on homology.

This theorem is due to Mather in the case $p=1$.

The map is certainly not a homotopy equivalence since $\pi_1(B \text{Diff}^r_K(R^p))$ is highly nonabelian while $\pi_1(\Omega^pB\Gamma^r_p) = \pi_{p+1}(B\Gamma^r_p)$ is abelian.

Similarly, there is a map $B \text{Diff}^r(M^p) \times M^p \to B\Gamma^r_p$ which is a lifting of the classifying map for the tangent bundle of $M^p$, so there is a commutative diagram

$$
\begin{array}{ccc}
B\Gamma^r_p & \longrightarrow & \text{BO}_p \\
\downarrow & & \\
B \text{Diff}^r(M^p) \times M^p & \longrightarrow & B\Gamma^r_p
\end{array}
$$

Let $X$ be the space of liftings of the classifying map for $T(M^p)$ in $\text{BO}_p$ to $B\Gamma^r_p$. Then we have a map $B \text{Diff}^r(M^p) \to X$.

**Theorem 5.** The map $B \text{Diff}^r(M^p) \to X$ induces an isomorphism on homology.

Again, this is not a homotopy equivalence since $\pi_1(X)$ is abelian.

For the case $r=0$, we assume $M^p$ is a differentiable manifold.

**Corollary.** (a) $B \text{Homeo}(M^p)$ is acyclic, where $\text{Homeo}(M^p) = \text{Diff}^0(M^p)$ is the group of homeomorphisms of $M^p$.

(b) The map $B \text{Homeo}(M^p) \to B \text{Homeo}(M^p)$ induces an isomorphism on homology.

This corollary is implied by Theorems 3 and 5. Cf. Mather [13], who showed $B \text{Homeo}_K(R^p)_\delta$ is acyclic.

**Corollary.** The following groups are isomorphic, where $k$ is the first positive integer such that one of them is nontrivial:

(i) $H_k(B \text{Diff}^r(M^p); Z)$,

(ii) $H_k(B \text{Diff}^r_K(R^p); Z)$,

(iii) $H_{k+p}(B\Gamma^r_p; Z)$.

**Conjecture.** This first $k$ is $p+1$, for $r=\infty$.

Mather’s theorem [11] shows this for $p=1$. Bott and Haefliger showed
that all differentiable characteristic classes (in some sense) vanish below this dimension, $H_{2p+1}(B\Gamma_p^\infty; \mathbb{Z})$ [2], [3].

In [16] I sketched examples showing there is a surjective homomorphism

$$H_2(B\Gamma_p^\infty; \mathbb{Z}) \twoheadrightarrow \mathbb{R},$$

using the Godbillon-Vey invariant $gv$ [5]. Recently I have extended this to arbitrary codimension, so there is a surjective homomorphism

$$H_{2p+1}(B\Gamma_p^\infty; \mathbb{Z}) \twoheadrightarrow \mathbb{R}.$$

BIBLIOGRAPHY


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