BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

POSITIVE NEAR-APPROXIMANTS AND SOME PROBLEMS OF HALMOS

BY RICHARD BOULDIN

Communicated by P. R. Halmos, July 27, 1973

In [5] P. R. Halmos called for an investigation of those nonnegative operators P with the property that the distance from P to a fixed operator T is the same as the distance of T to the set of nonnegative operators. Such a P is a "positive approximant" of T. Halmos asked for the properties of such best approximating nonnegative operators when other norms besides the operator norm were used to compute distance. If $T=B+iC$, with $B=B^*$, $C=C^*$, then the formula $|||T|||^2=\|B^2+C^2\|$ defines a norm on the bounded operators with the pıoperty that

$$\|T\| \geqq |||T||| \geqq w(T) \geqq \tfrac{1}{2}\,\|T\|$$

where $w(T)$ denotes the numerical radius of T. The distance from T to the nonnegative operators is the same whether it is computed with the operator norm or with the new norm. A nonnegative operator which best approximates T in the new norm is a "positive near-approximant." This name is motivated by the facts that every positive approximant is a positive near-approximant and a positive near-approximant frequently turns out to be a positive approximant, although that is not necessarily the case.

P. R. Halmos gave an ingenious argument which resulted in a device for computing the distance of T to the nonnegative operators, denoted $\delta(T)$, and in a formula which defines a positive approximant of T for any T. If $T=B+iC$, with $B=B^*$, $C=C^*$, then the Halmos positive approximant is $P_0=B+(\delta^2-C^2)^{1/2}$ where $\delta=\delta(T)$. In [1] we showed that P_0 is absolutely maximal for the positive approximants of T, that is $P\leqq P_0$ whenever $P\in\mathscr{P}(T)$ with $\mathscr{P}(T)$ denoting the positive approximants; we used this fact as a basis for constructing positive approximants. In [2] we showed that P_0 is absolutely maximal for the positive near-approximants of T, denoted $\mathscr{P}'(T)$, and from this we constructed positive near-approximants. We have now carried this approach to the point of

AMS (MOS) subject classifications (1970). Primary 47A55; Secondary 46B99.
Key words and phrases. Nonnegative operator, best approximation, normal operator, positive approximant, convex, extreme point.

determining much of the convex structure of the two convex sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$, and it is these results which are described in this announcement. We denote the underlying complex Hilbert space by H and the range of the operator A is written AH.

THEOREM 1. *Let $\mathscr{P}'(T)$ denote the convex set of positive near-approximants of the normal operator T and let H_0 denote the subspace*

$$(P_0H)^- \cap ((\delta^2 - C^2)H)^-.$$

If the dimension of H_0 is denoted by p then we have

$$\dim \mathscr{P}'(T) = p^2.$$

Here all infinite cardinal numbers are identified.

COROLLARY 1. *The normal T has a unique positive near-approximant if and only if the closed span of the kernels of P_0 and of $(\delta^2 - C^2)$ is all of H.*

COROLLARY 2. *Let $\mathscr{P}(T)$ denote the convex set of positive approximants of T and let p denote the dimension of H_0. Assume that either* (i) *or* (ii) *below holds*:

(i) *T is normal,*

(ii) *C and C^2 have the same commutant where $T = B + iC$ with $B = B^*$, $C = C^*$.*

Then we have $\dim \mathscr{P}'(T) = \dim \mathscr{P}(T) = p^2$.

In the event that T is normal the distance $\delta(T)$ is given by a simple formula, namely $\delta(T) = \|B_- + iC\|$, where $T = B + iC$, $B = B^*$, $C = C^*$ and $2B_- = (B^2)^{1/2} - B$. Halmos first proved this with a matrix argument. In **[1]** we showed that T has a unique positive approximant if and only if every point of the spectrum of T has a distance of δ to the nonnegative reals. We can now give a very quick proof of both of these facts. Let $\mathscr{A}$ denote the C^* algebra generated by T and let $C(\sigma(T))$ denote space of continuous functions on the spectrum of T with the usual norm. The Gelfond transform, denoted Γ, gives an isometric isomorphism of $\mathscr{A}$ onto $C(\sigma(T))$ and $(\Gamma T)(z) = z$. (See paragraphs 4.30 and 4.31, pp. 93–94, of **[4]**.) The positive part of the real part of T, denoted B_+, and the Halmos positive approximant belong to $\mathscr{A}$ and clearly the corresponding continuous functions are $\varphi(z) = x$ and $\psi(z) = x + (\delta^2 - y^2)^{1/2}$, respectively, where $z = x + iy$ with x and y real. By considering the right and left half planes separately one easily sees that

$$\|z - p(z)\|^2 = \text{supremum}\{(x_-)^2 + y^2 : z \in \sigma(T)\}$$

where $p(z)$ is any nonnegative valued function continuous on $\sigma(T)$. By the spectral mapping theorem and the observation that $B_- + iC$ is a normal operator we obtain $\|z-p(z)\| \geqq \|B_- + iC\|$. From this inequality it is immediate that $\delta(T) = \|B_- + iC\|$ and B_+ is a positive approximant for T.

One easily verifies that the distance from z to the nonnegative reals is d if and only if we have

$$(*) \qquad -x_- + (d^2 - y^2)^{1/2} = 0.$$

Since B_+ is a positive approximant of T and $\sigma(T)$ is compact there is a point z_0 in $\sigma(T)$ such that

$$\delta = |z_0 - \varphi(z_0)| = |-(x_0)_- + iy_0|.$$

Consequently, if every point of $\sigma(T)$ has a distance of d to the nonnegative reals then $d = \delta$. One now sees that $(*)$ above is equivalent to saying that $\varphi(z)$ and $\psi(z)$ coincide, or B_+ and P_0 coincide. We have given an elementary argument in [**2**] to show that this implies

$$\mathscr{P}'(T) = \mathscr{P}(T) = \{P_0\}.$$

On the other hand, if T has a unique positive approximant then P_0 equals B_+ and $\varphi(z)$ coincides with $\psi(z)$. This is equivalent to the assertion that every point of $\sigma(T)$ has distance δ to the nonnegative reals.

The very commutative approach of the above two paragraphs produces very little beyond the above proofs. The one further fact that can be deduced by this approach is that $\mathscr{P}(T)$, for T a normal operator, is a finite dimensional convex set only if $\sigma(T)$ has only finitely many points with distance to the nonnegative reals not equal to δ. In contrast the very elaborate constructive approach which produced Theorem 1 and its corollaries yields further information. An important preliminary fact is that $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ are both compact in the weak operator topology and consequently each is the closed convex hull of its extreme points by the Krein-Milman Theorem.

THEOREM 2. *Assume that T is normal and that H_0 (as defined in Theorem 1) is finite dimensional. Define the operator A_0 by the formula*

$$2A_0 = P_0 + R - |P_0 - R|$$

where $R = 2(\delta^2 - C^2)^{1/2}$. Then A_0, P_0 and C commute and each is reduced by H_0. Let $\{e_1, \cdots, e_p\}$ be an orthonormal basis for H_0 which simultaneously diagonalizes the restrictions of A_0, P_0 and C; let Q be the orthogonal projection onto e_j for some $j = 1, \cdots, p$. Then $P_0 - A_0Q$ is an extreme point of each of the sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$.

COROLLARY. *Assume that T is normal and that H_0 is a one dimensional subspace. Let f_0 be a unit vector in H_0 and let λ_0 and A_1 be defined by the* *equations*

$$\lambda_0 = \min\{\langle P_0 f_0, f_0\rangle, \langle 2(\delta^2 - C^2)^{1/2} f_0, f_0\rangle\}, \qquad A_1 = \langle \cdot, f_0\rangle \lambda_0 f_0.$$

Then $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ coincide with the convex hull of P_0 and $P_0 - A_1$; consequently we have

$$\mathscr{P}'(T) = \mathscr{P}(T) = \{P_0 - \lambda A_1 : \lambda \in [0, 1]\}.$$

The detailed proofs of these results will appear elsewhere.

References

1. R. H. Bouldin, *Positive approximants,* Trans. Amer. Math. Soc. **177** (1973), 391–403.

2. ———, *Operators with a unique positive near-approximant*, Indiana Univ. Math. J. (to appear).

3. ———, *The convex structure of the set of positive approximants for a given operator* (to appear).

4. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

5. P. R. Halmos, *Positive approximants of operators*, Indiana Univ. Math. J. **21** (1971/72), 951–960. MR **45** #919.

Department of Mathematics, University of Georgia, Athens, Georgia 30601