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THE SPECTRAL MAPPING THEOREM FOR JOINT APPROXIMATE POINT SPECTRUM

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ABSTRACT. The spectral mapping theorem for joint approximate point spectrum is proved when A is an n-tuple of commuting operators on a Banach space and f is any m-tuple of rational functions for which $f(A)$ is defined.

The purpose of the paper is to show how properties of parts of the joint spectrum can be obtained by use of spaces of sequences of vectors. The method (first used for general Banach spaces in [**4**]) provides very simple proofs of some known results; but the main result is believed to be new.

Throughout the paper, we deal with n-tuples of bounded operators on a Banach space $\mathscr{X}$, whose vectors are denoted by $x, y, \cdots$. It will be convenient to use a symbol such as A to denote an n-tuple of operators and a symbol such as λ to denote an n-tuple of complex numbers: $A=(A_1, \cdots, A_n)$, $\lambda=(\lambda_1, \cdots, \lambda_n)$, $0=(0, \cdots, 0)$. The one-sided spectra of interest in this paper are not defined in terms of existence of Banach-algebra inverses, the context of the papers of Robin Harte ([**5**], [**6**]); they are the somewhat different spectra of the following definition. (However, in the special case that $\mathscr{X}$ is a Hilbert space, our result is a corollary of Harte's.)

DEFINITION. *We say* $0 \in \sigma_p(A)$ *in case there exists a nonzero* $x \in \mathscr{X}$ *such that* $Ax=0$ (*i.e.*, $A_jx=0$ *for* $j=1, \cdots, n$). *We say* $0 \in \sigma_\pi(A)$ *in case for every* $\varepsilon>0$ *there exists a unit vector* $x \in \mathscr{X}$ *such that* $\|Ax\|<\varepsilon$ (*i.e.*, $\|A_jx\|<\varepsilon$ *for* $j=1, \cdots, n$). *We say* $0 \in \sigma_\delta(A_1, \cdots, A_n)$ *in case* $0 \in \sigma_\pi(A_1^*, \cdots, A_n^*)$, *where* A_j^* *denotes the Banach-space adjoint of* A_j. *We say* $\lambda \in \sigma_p(A)$ *in case* $0 \in \sigma_p(A-\lambda)$, *and similarly for* σ_π, σ_δ. *The set* $\sigma_p(A)$ *is called the 'point spectrum' of the n-tuple* A; *the set* $\sigma_\pi(A)$, *its 'approximate point spectrum' or 'left approximate spectrum'; the set* $\sigma_\delta(A)$, *its 'approximate defect spectrum' or 'right approximate spectrum'.*

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Property 1. *In order that* $0 \notin \sigma_\delta(A)$ *it is necessary and sufficient that* $\sum_j \text{range}(A_j)=\mathscr{X}$.

This is well known for A a single operator [**10**, §4.7], even in the more general case that A has as its domain a Banach space different from $\mathscr{X}$. But the statement for an n-tuple $A=(A_1, \cdots, A_n)$ is just another special case of this: consider such A as an operator from $\mathscr{X}\oplus\cdots\oplus\mathscr{X}$ to $\mathscr{X}$.

Property 2. *If A is an n-tuple of commuting operators, then* $\sigma_\pi(A)$ *is a nonvoid compact set.* (*Hence so is* $\sigma_\delta(A)$.)

This too is well known for the case of a single operator. The compactness is easy to prove in general. That σ_π is nonvoid for a commuting n-tuple will follow from our main result; see the discussion of Property 4 below.

We rely on what we call the 'Berberian-Quigley extension', an adaptation of ideas of F. D. Quigley [**9**, pp. 25–26] and S. K. Berberian [**1**]. Denote by $\boldsymbol{m}(\mathscr{X})$ the vector space of bounded sequences $(x_\nu)_{\nu=1}^\infty$, $x_\nu \in \mathscr{X}$; by $\boldsymbol{n}(\mathscr{X})$, the subspace of null sequences (those such that $\|x_\nu\|\to 0$). If we set $\|(x_\nu)\|=\lim\sup\|x_\nu\|$ for every sequence (x_ν), this defines a seminorm on $\boldsymbol{m}(\mathscr{X})$, which is zero exactly on the elements of $\boldsymbol{n}(\mathscr{X})$. Indeed, as was pointed out to us by M. Wolff, it agrees with the usual norm on the quotient space $\boldsymbol{m}(\mathscr{X})/\boldsymbol{n}(\mathscr{X})$. This space, which is therefore complete, will be our extension space $\mathscr{X}^0$. Given any operator A on $\mathscr{X}$, we observe that mapping (x_ν) to (Ax_ν) is a tranformation taking $\boldsymbol{m}(\mathscr{X})$ to $\boldsymbol{m}(\mathscr{X})$ and $\boldsymbol{n}(\mathscr{X})$ to $\boldsymbol{n}(\mathscr{X})$, indeed that $\|(Ax_\nu)\|\leqq\|A\|\ \|(x_\nu)\|$. It is easy to conclude that this mapping has a unique extension $A^0:\mathscr{X}^0\to\mathscr{X}^0$, with $\|A^0\|=\|A\|$. This construction seems simpler than Hirschfeld's [**8**] original version.

PROPOSITION 1. *The correspondence* $A\to A^0$ *is an injective homomorphism, and* $\sigma_\pi(A)=\sigma_\pi(A^0)=\sigma_p(A^0)$.

As in [**1**], [**4**], these are the properties needed for our application. For this reason, we set forth the essentially familar proof. The homomorphic property is immediate, as is the inclusion $\sigma_p(A^0)\subseteq\sigma_\pi(A^0)$. To prove $\sigma_\pi(A)\subseteq\sigma_p(A^0)$, it is sufficiently general to assume $0\in\sigma_\pi(A)$—hence to assume there exists, for each natural number ν, a unit vector $x_\nu\in\mathscr{X}$ such that $\|Ax_\nu\|<1/\nu$—and deduce that A^0 has a nontrivial null-vector. Now the sequence (x_ν) formed from the hypothesized vectors is in $\boldsymbol{m}(\mathscr{X})$, and $\|(x_\nu)\|=\lim\|x_\nu\|=1$ while $\|(Ax_\nu)\|=\lim\|Ax_\nu\|=0$; hence the image of (x_ν) in $\mathscr{X}^0$ is the null-vector sought.

To prove $\sigma_\pi(A^0)\subseteq\sigma_\pi(A)$, it is sufficient to assume $0\in\sigma_\pi(A^0)$—hence to assume there exists, for each ν, a unit vector $\boldsymbol{x}_\nu\in\mathscr{X}^0$ such that $\|A^0\boldsymbol{x}_\nu\|<1/\nu$—and deduce a similar property of A. Each $\boldsymbol{x}_\nu$ is in $\boldsymbol{m}(\mathscr{X})/\boldsymbol{n}(\mathscr{X})$; and then any sequence in $\boldsymbol{m}(\mathscr{X})$ which represents $\boldsymbol{x}_\nu$ must contain a vector y_ν

arbitrarily close to 1 in norm and with $\|Ay_\nu\|$ arbitrarily close to $\|A^0\boldsymbol{x}_\nu\|$. The conclusion now clearly follows.

We have arranged the definition and notations so that the above Proposition applies without change to the case of n-tuples. Notice that, unlike the theorems below, it is unaffected by the possibility that the A_j not commute.

It may be worth digressing to explain the relationship of the Berberian-Quigley extension to the Berberian extension [**1**]. For this digression only, assume $\mathscr{X}$ a Hilbert space. The Berberian-Quigley extension is still not a Hilbert space (even if $\mathscr{X}$ is the 1-dimensional space $\boldsymbol{C}$); thus if $\|x_\nu\|=1+(-1)^\nu$ and $\|y_\nu\|=1-(-1)^\nu$ then (x_ν) and (y_ν) fail to satisfy the parallelogram identity. Berberian works from an arbitrary generalized limit, which we here write L. That is, L is an arbitrary linear functional on $\boldsymbol{m}(\boldsymbol{C})$ with the property that for any real sequence (α_ν), $L((\alpha_\nu))\leqq\limsup\alpha_\nu$. (No translation-invariance requirement is imposed upon L.) Setting $\|(x_\nu)\|_L=(L(\|x_\nu\|^2))^{1/2}$ gives a seminorm on $\boldsymbol{m}(\mathscr{X})/\boldsymbol{n}(\mathscr{X})$ (thence also on $\mathscr{X}^0$) which *is* derived from an inner product. Berberian, using this seminorm, factors out a larger null-space than our $\boldsymbol{n}(\mathscr{X})$ and ends with a "smaller" extension space which is a Hilbert space. The relation between the two norms is

$$\|(x_\nu)\|^2 = \limsup_\nu \|x_\nu\|^2 = \sup_L L(\|x_\nu\|^2) = \sup_L \|(x_\nu)\|_L^2.$$

We proceed to the applications of the method.

Property 3. *For every polynomial f, and for a single operator A, $\sigma_\pi(f(A))=f(\sigma_\pi(A))$.*

Property 4. *Let f be the m-tuple of functions ($m<n$) defined by $f_1(\lambda)=\lambda_1,\cdots,f_m(\lambda)=\lambda_m$. Let A be an n-tuple of commuting operators. Then $\sigma_\pi(f(A))=f(\sigma_\pi(A))$. (In particular, this implies that σ_π of an n-tuple is nonvoid.)*

Property 3 is in [**7**, §5.12]; Property 4 for the case of a Hilbert space would be a corollary of results of Bunce [**2**]; cf. Żelazko [**12**]. These two facts are both special cases of the following new theorem, of which we now give an independent proof.

THEOREM 1. *Let A be an n-tuple of commuting operators. Let f be an m-tuple of polynomials in n variables (so that $f(A)$ is defined and is an m-tuple of commuting operators). Then $\sigma_\pi(f(A))=f(\sigma_\pi(A))$. Hence the same holds for σ_δ.*

That $f(\sigma_\pi(A))\subseteq\sigma_\pi(f(A))$ follows directly from definitions. For the other direction, assume $\lambda\in\sigma_\pi(f(A))$. Then by Proposition 1, $\lambda\in\sigma_p(f(A)^0)$, and $f(A)^0=f(A^0)$. Let $\mathscr{M}$ be the nontrivial subspace of null-vectors of

$f(A)^0-\lambda$. It is invariant under A^0 (i.e., under each A_j^0). Choose any μ_1 in the nonvoid set $\sigma_\pi(A_j^0|_{\mathscr{M}})$. Then from the definition and Proposition 1, the $(m+1)$-tuple (λ, μ_1) belongs to $\sigma_\pi(f(A)^0|_{\mathscr{M}}, A_1^0|_{\mathscr{M}})\subseteq\sigma_\pi(f(A)^0, A_1^0)=\sigma_\pi(f(A), A_1)$. Repeat the above process, using in the role of $f(A)$ first $(f(A), A_1)$, then $(f(A), A_1, A_2)$, and so forth. The result is an n-tuple $\mu=(\mu_1, \cdots, \mu_n)$ such that $(\lambda, \mu)\in\sigma_\pi(f(A), A)$. Thus $\mu\in\sigma_\pi(A)$, as we can see directly by the definition. Employing Berberian-Quigley extension again, we see that there exists a nonzero $\boldsymbol{x}$ in the extension space $\mathscr{X}^0$ such that $f(A^0)\boldsymbol{x}=\lambda\boldsymbol{x}$ and $A^0\boldsymbol{x}=\mu\boldsymbol{x}$. This leads to $\lambda\boldsymbol{x}=f(\mu)\boldsymbol{x}$, $\lambda=f(\mu)$, so that $\lambda\in f(\sigma_\pi(A))$ as required.

COROLLARY. *If all A_j and B_k commute, then $\sigma_\pi(A+B)\subseteq\sigma_\pi(A)+\sigma_\pi(B)$.*

Here f must be defined by $f(\lambda, \mu)=\lambda+\mu$ (where we still allow λ, μ, A, B to be n-tuples). Then $\sigma_\pi(A+B)$ is $\sigma_\pi(f(A, B))$, which by the theorem is $f(\sigma_\pi(A, B))$. This in turn is $\subseteq f(\sigma_\pi(A), \sigma_\pi(B))$ just because $\sigma_\pi(A,B)\subseteq\sigma_\pi(A)\times\sigma_\pi(B)$, which is essentially the trivial half of Property 4.

The corollary, in the case $n=1$, was the part of this subject which was applied in [**4**].

Of course we want also to consider the spectral mapping theorem for more general functions. For rational $f=g/h$ (g and h polynomials), one expects such a result to hold provided $f(A)$ is defined; one expects, accordingly, to assume h has no zeros on the joint spectrum of A. Several definitions of joint spectrum recommend themselves [**11**], but this does not concern us here. The formally weakest restriction will be to exclude zeros of h from the smallest possible set. We adopt $\sigma_\pi\cup\sigma_\delta$, which is contained properly [**3**] in any reasonable joint spectrum.

THEOREM 2. *Let A be an n-tuple of commuting operators. Let f be an m-tuple of rational functions, $f_i=g_i/h_i$, and assume each h_i has no zeros on $\sigma_\pi(A)\cup\sigma_\delta(A)$. Then $\sigma_\pi(f(A))=f(\sigma_\pi(A))$. Hence the same holds for σ_δ.*

The only substantial issue is the definition of f. Because h has no zeros in $\sigma_\pi(A)$ (i.e., $0\notin h(\sigma_\pi(A))$), we know from Theorem 1 that $0\notin\sigma_\pi(h(A))$. Similarly, because h has no zeros in $\sigma_\delta(A)$, we know from Theorem 1 that $0\notin\sigma_\delta(h(A))$. Thus each $h_i(A)$ is invertible; hence there is no difficulty in defining $f(A)$. This, with the homomorphism property of the Berberian-Quigley extension, is all that is required to imitate the reasoning of Theorem 1, completing the proof of Theorem 2.

COROLLARY. *If we define σ by $\sigma(A)=\sigma_\pi(A)\cup\sigma_\delta(A)$, then we have, for every commuting n-tuple A of operators and for every m-tuple f of* *rational functions* $f_i = g_i/h_i$ *such that each* h_i *has no zeros on* $\sigma(A)$, *that* $\sigma(f(A)) = f(\sigma(A))$.

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Added in proof. We have learned that our theorems were obtained independently by a quite different method by Żelazko and Słodkowski in a paper, forthcoming in Studia Math., which we have not had the opportunity to see.

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