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COEFFICIENTS FOR ALPHA-CONVEX UNIVALENT FUNCTIONS

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Let α be a nonnegative real number, and let $M(\alpha)$ denote the class of normalized α-convex univalent functions f in the open unit disc $E=\{z:|z|<1\}$, i.e., $f\in M(\alpha)$ if and only if f is regular in E, $f(0)=f'(0)-1=0$, $f(z)f'(z)/z\neq 0$ for $z\in E$, and

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]\right\}>0$$

for $z\in E$ [3], [4]. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, the coefficient bounds for $|a_2|$ and $|a_3|$ are known [2], [4]; an inequality relating the coefficients $|a_n|$ for $n=2,3,\cdots$ is found in [2]; yet the determination of the coefficient bound for $|a_n|$ has so far been an open problem.

Here we announce the general result for this coefficient problem; the proof will be published elsewhere.

THEOREM. *Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in M(\alpha)$. Let $S(n)$ be the set of all n-tuples $(r_1,r_2,\cdots,r_n)$ of nonnegative integers for which $r_1+2r_2+3r_3+\cdots+nr_n=n$, and for each such n-tuple define m by $r_1+r_2+\cdots+r_n=m$. If $\gamma(\alpha,m)=\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-m)$ with $\gamma(\alpha,0)=1$, then for $n=1,2,\cdots$*

$$|a_{n+1}|\leqq\sum\frac{\gamma(\alpha,m-1)c_1^{r_1}c_2^{r_2}\cdots c_n^{r_n}}{r!\,r_2!\cdots r_n!},\tag{1}$$

where summation is taken over all n-tuples in $S(n)$, and

$$c_n=\frac{2(2+\alpha)(2+2\alpha)\cdots[2+(n-1)\alpha]}{n!\,\alpha^n(1+n\alpha)}.$$

The bounds in (1) are sharp and for $\alpha>0$ attained by

$$f(z)=\left[\frac{1}{\alpha}\int_0^z\zeta^{1/\alpha-1}(1-\zeta)^{-2/\alpha}\,d\zeta\right]^{\alpha}.$$

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For $\alpha=0$, we find from (1) that $|a_n|\leqq n$ for $n=2, 3, \cdots$ and the bounds are attained by the function $f(z)=z(1-z)^{-2}$. For $\alpha=1$, $|a_n|\leqq 1$ for $n=2, 3, \cdots$ the bounds being attained by $f(z)=z(1-z)^{-1}$.

The technique used by Goodman in [**1**] has been employed to get the bounds in (1) in the compact form.

Thus, it is easy to find from (1) that, e.g.,

$$|a_2| \leqq 2/(1+\alpha),$$
$$|a_3| \leqq (3+8\alpha+\alpha^2)/(1+\alpha)^2(1+2\alpha),$$
$$|a_4| \leqq 4(3+19\alpha+38\alpha^2+11\alpha^3+\alpha^4)/3(1+\alpha)^3(1+2\alpha)(1+3\alpha),$$
$$|a_5| \leqq \frac{30+394\alpha+2024\alpha^2+5284\alpha^3+6386\alpha^4+2638\alpha^5+488\alpha^6+36\alpha^7}{6(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}.$$

The formula (1) is however readily computable.

It may be noted that $\sup|a_{n+1}|<\sup|a_n|$ for $\alpha\geqq 2$, $n=2, 3, \cdots$. Also, for a given n, $n=2, 3, \cdots$, $\sup|a_n|$ is a decreasing function of α, $\alpha\geqq 0$.

References

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