GENERALIZED SUPER-PARABOLIC FUNCTIONS

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The purpose of this note is to announce results which generalize potential theory (superharmonic functions) to a broad class of parabolic operators. Many of the properties of superharmonic functions carry over to functions in this new class. Let \( Q = \Omega \times (0, T) \) where \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( T > 0 \) is a scalar. All functions will be defined on \( \overline{Q} \) and will be written as functions of \( (x, t) \) with \( x \in \bar{\Omega} \) and \( t \in [0, T] \).

For \( (x, t) \in \overline{Q} \) assume

(a) \( a_{ij}(x, t) \) is a bounded, measurable function for \( i, j = 1, 2, \cdots, n \) and assume there is a constant \( \lambda > 0 \) such that \( \sum a_{ij}(x, t)z_iz_j \geq \lambda |z|^2 \) for all \( z \in \mathbb{R}^n \) and almost all \( (x, t) \in Q \).

(b) \( c(x, t) \in L^q[0, T; L^p(\Omega)] \) for \( n/2p + 1/q < 1, 1 < p, q < \infty \).

(c) \( b_j(x, t), d_j(x, t) \in L^q[0, T; L^p(\Omega)] \) for \( j = 1, \cdots, n \) and \( n/2p + 1/q < \frac{1}{2}, 2 < p, q < \infty \).

The parabolic operator under consideration is defined by

\[
Lu = u_t - \sum a_{ij}(x, t)u_{x_i} - \sum b_j(x, t)u_x - c(x, t)u
\]

where \( u_x = \frac{\partial u}{\partial x_j} \) and an index \( i \) or \( j \) is summed over \( 1 \leq i, j \leq n \) whenever it is repeated in a product.

**Definition 1.** \( u(x, t) \) is a weak solution of \( Lu = 0 \) in \( Q \) if \( u \) is locally in \( L^2[0, T; H^{1,2}(\Omega)] \) and \( \int_Q \left[ a_{ij}u_{x_i}\phi_{x_j} + b_ju_x\phi_x - c(u_x)\phi - cu\phi - u\phi_t \right] dx \, dt = 0 \) for all \( \phi \in C^1(\overline{Q}) \).

Let \( \partial_p Q = \{ \partial_\Omega \times [0, T] \} \cup \{ \Omega \times (0) \} \) denote the parabolic boundary of \( Q \). Due to the number of definitions and results, they are stated below with no proofs.

**Theorem 1.** Let \( f \in C(\partial_p Q) \) and let \( u = u(x, t) \) be the weak solution of the boundary value problem

\[
Lu = 0 \quad \text{on} \quad Q, \quad u = f \quad \text{on} \quad \partial_p Q.
\]

Then, to each \( (x, t) \in Q \), there corresponds a nonnegative Borel measure

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\[
\mu_{(x, t)} \text{ on } \partial_x Q \text{ such that}
\]

\[
u(x, t) = \int_{\partial x Q} f \, d\mu_{(x, t)} \text{ on } Q.
\]

In the future, write \( L(f; (x, t), Q) = \int_{x \in Q} f \, d\mu_{(x, t)} \).

**Definition 2.** \( u \in S_Q \) if and only if \( u \in L^a[0, T; H^{1,2}(\Omega)] \) and for all \( \phi \in C_0^1(\Omega^3) \) with \( \phi \geq 0 \),

\[
\int_Q \left[ a_{ij} u_{,j} \phi_{,i} + d_{ij} \phi_{,j} u - b_{ij} u_{,j} \phi - cu\phi - u\phi_i \right] \, dx \, dt \geq 0.
\]

**Definition 3.** \( R_a(x_0, t_0) = \{(x, t); |x_i - x_{0i}| < a, t_0 - a^2 < t \leq t_0\} \) is called a standard rectangle based at \((x_0, t_0)\).

**Definition 4.** \( u \in l(Q) \) if and only if

(i) \( u \geq + \infty \) on \( Q \),

(ii) \( u > - \infty \) on \( Q \), and

(iii) \( u \) is lower semicontinuous on \( Q \).

**Definition 5.** The extended real valued Borel measurable function \( u \) defined on an open set \( D \) is

(a) super-mean-valued at \( z \in D \) if \( L(u; z, R_\delta) \) is defined and \( u(z) \geq L(u; z, R_\delta) \) for almost all \( \delta \) with \( R_\delta(z) \subset D \);

(b) super-mean-valued on \( D \) if it is super-mean-valued at each \( z \in D \);

(c) locally super-mean-valued at \( z \in D \) if there is a \( \delta(z) > 0 \) such that \( R_{\delta(z)}(z) \subset D \) and \( u(z) \geq L(u; z, R_\delta) \) for all \( \delta < \delta(z) \);

(d) locally super-mean-valued on \( D \) if it is locally super-mean-valued at each \( z \in D \).

**Definition 6.** \( S'_Q = \{ u \in l(Q); u \) is super-mean-valued on \( Q \} \). \( S''_Q = \{ u \in l(Q); \) for any cylinder \( W = C \times (a, b) \) with \( W \subset Q \), and any \( v \) with \( v \in C(W), Lv = 0 \) on \( W \), and \( u \geq v \) on \( \partial_p W \), it follows that \( u \geq v \) on \( W \} \). \( S'''_Q = \{ u \in l(Q); u \) is locally super-mean-valued on \( D \}. \)

**Theorem 2.** \( u \in S_Q \) with \( u \geq 0 \) on \( \partial_p Q \) implies \( u \geq 0 \) on \( Q \).

**Corollary.** If \( c + \{d_{ij}\}_{ij} \leq 0 \) weakly on \( Q \), then the weak solution \( u \) of \( Lu = 0 \) in \( Q \), \( u = 1 \) on \( \partial_p Q \) satisfies \( 0 \leq u(x, t) \leq 1 \) on \( Q \).

From now on assume \( c + \{d_{ij}\}_{ij} \leq 0 \) weakly on \( Q \).

**Theorem 3.** Let \( u \in S''_Q \). If, for some \( (x_0, t_0) \in Q \), \( u(x_0, t_0) = \inf_Q u \leq 0 \), then \( u(x, t) \equiv u(x_0, t_0) \) on \( \Omega \times (0, t_0) \).

**Theorem 4.** \( S_Q \subset S'_Q = S''_Q = S'''_Q \).

**Theorem 5.** Let \( F(x) \) be convex on \( \mathbb{R}^n \) with \( F(0) \leq 0 \). If \( Lu = 0 \) on \( Q \), then \( -F(u) \in S'_Q \).

**Theorem 6.** Let \( F(x) \) be nondecreasing and convex on \( \mathbb{R}^n \) with \( F(0) \leq 0 \). If \( -u \in S'_Q \), then \( -F(u) \in S'_Q \).
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THEOREM 7. If \( u \in S'_Q \), and if \( u(x, t) \geq 0 \), then there exist \( t_0, t_1 \) with \( 0 \leq t_0 \leq t_1 \leq T \) such that

\[
\begin{align*}
    u(x, t) &\equiv 0 & &\text{on } \Omega \times (0, t_0), \\
    0 < u(x, t) &< +\infty & &\text{on } \Omega \times (t_0, t_1), \\
    u(x, t) &\equiv +\infty & &\text{on } \Omega \times (t_1, T).
\end{align*}
\]

THEOREM 8. If \( u, v \in S'_Q \) and \( c > 0 \), then (i) \( cu \in S'_Q \), (ii) \( u + v \in S'_Q \), (iii) \( \inf(u, v) \in S'_Q \).

THEOREM 9. \( u, -u \in S'_Q \) implies \( Lu = 0 \) weakly on \( Q \).

THEOREM 10. Let \( u \in S'_Q \) and let \( R \) be a standard rectangle with \( \bar{R} \subset Q \). Set

\[
\begin{align*}
    v(x, t) &= L(u; (x, t), R) & &\text{if } (x, t) \in R, \\
    &= u(x, t) & &\text{if } (x, t) \in Q - R.
\end{align*}
\]

Then \( u \preceq v \) on \( Q \), \( Lv = 0 \) on \( R \), and \( v \in S'_Q \).

REFERENCES


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