GENERALIZED SUPER-PARABOLIC FUNCTIONS

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Communicated by Alberto Calderón, September 22, 1973

The purpose of this note is to announce results which generalize potential theory (superharmonic functions) to a broad class of parabolic operators. Many of the properties of superharmonic functions carry over to functions in this new class. Let \( Q = \Omega \times (0, T) \) where \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( T > 0 \) is a scalar. All functions will be defined on \( \bar{Q} \) and will be written as functions of \( (x, t) \) with \( x \in \bar{\Omega} \) and \( t \in [0, T] \).

For \( (x, t) \in \bar{Q} \) assume

(a) \( a_{ij}(x, t) \) is a bounded, measurable function for \( i, j = 1, 2, \ldots, n \) and assume there is a constant \( \lambda > 0 \) such that \( \sum a_{ij}(x, t)z_i z_j \geq \lambda |z|^2 \) for all \( z \in \mathbb{R}^n \) and almost all \( (x, t) \in Q \).

(b) \( c(x, t) \in L^p[0, T; L^q(\Omega)] \) for \( n/2p + 1/q < 1, 1 < p, q \leq \infty \).

(c) \( b_j(x, t), d_j(x, t) \in L^{2p/2p+1/q}[0, T; L^q(\Omega)] \) for \( j = 1, \ldots, n \) and \( n/2p + 1/q < \frac{1}{2} \).

The parabolic operator under consideration is defined by

\[
Lu = u_t - \{a_{ij}(x, t)u_{x_i} + d_j(x, t)u_{x_j} - b_j(x, t)u - c(x, t)u\}
\]

where \( u_{x_j} = \frac{\partial u}{\partial x_j} \) and an index \( i \) or \( j \) is summed over \( 1 \leq i, j \leq n \) whenever it is repeated in a product.

**DEFINITION 1.** \( u(x, t) \) is a weak solution of \( Lu = 0 \) in \( Q \) if \( u \) is locally in \( L^2[0, T; H^{1,2}(\Omega)] \) and \( \int_Q [a_{ij}u_{x_j} \phi_{x_j} + d_j u \phi - b_j u \phi - cu \phi - u \phi_t] \, dx \, dt = 0 \) for all \( \phi \in C_0^1(\bar{Q}) \).

Let \( \partial_p Q = \{\partial \Omega \times [0, T]\} \cup \{\Omega \times (0)\} \) denote the parabolic boundary of \( Q \). Due to the number of definitions and results, they are stated below with no proofs.

**THEOREM 1.** Let \( f \in C(\partial_p Q) \) and let \( u = u(x, t) \) be the weak solution of the boundary value problem

\[
Lu = 0 \quad \text{on} \quad Q, \quad u = f \quad \text{on} \quad \partial_p Q.
\]

Then, to each \( (x, t) \in Q \), there corresponds a nonnegative Borel measure

\( AMS (MOS) \) subject classifications (1970). Primary 35K20, 31C05; Secondary 35D05.

Key words and phrases. Superharmonic functions, parabolic operators.

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\[ \mu_{(x,t)} \text{ on } \partial_p Q \text{ such that} \]
\[ u(x, t) = \int_{\partial_p Q} f \, d\mu_{(x,t)} \text{ on } Q. \]

In the future, write \( L(f; (x, t), Q) = \int_{\partial_p Q} f \, d\mu_{(x,t)} \).

**Definition 2.** \( u \in S_Q \) if and only if \( u \in L^a[0, T; H^{1,2}(\Omega)] \) and for all \( \phi \in C_0^\infty(Q^t) \) with \( \phi \geq 0 \),
\[ \int_Q \left[ a_{ij} u_{x_i x_j} + d_j \phi_j + u - b_j u \phi - cu \phi - u \phi_i \right] \, dx \, dt \geq 0. \]

**Definition 3.** \( R(a, t_0) = \{(x, t); |x - x_0| < a, t_0 - a^2 < t < t_0\} \) is called a standard rectangle based at \((x_0, t_0)\).

**Definition 4.** \( u \in l(Q) \) if and only if
(i) \( u \neq +\infty \) on \( Q \),
(ii) \( u > -\infty \) on \( Q \), and
(iii) \( u \) is lower semicontinuous on \( Q \).

**Definition 5.** The extended real valued Borel measurable function \( u \) defined on an open set \( D \) is
(a) super-mean-valued at \( z \in D \) if \( L(u; z, R_\delta) \) is defined and \( u(z) \geq L(u; z, R_\delta) \) for almost all \( \delta \) with \( R_\delta \subset D \);
(b) super-mean-valued on \( D \) if it is super-mean-valued at each \( z \in D \);
(c) locally super-mean-valued at \( z \in D \) if there is a \( \delta(z) > 0 \) such that \( R_{\delta(z)} \subset D \) and \( u(z) \geq L(u; z, R_\delta) \) for all \( \delta < \delta(z) \);
(d) locally super-mean-valued on \( D \) if it is locally super-mean-valued at each \( z \in D \).

**Definition 6.** \( S_Q' = \{u \in l(Q); u \text{ is super-mean-valued on } Q\} \). \( S_Q'' = \{u \in l(Q); \text{ for any cylinder } W = C \times (a, b) \text{ with } \overline{W} \subset Q, \text{ and any } v \text{ with } v \in C(\overline{W}), L_\nu = 0 \text{ on } W, \text{ and } u \geq v \text{ on } \partial_p W, \text{ it follows that } u \geq v \text{ on } W\}. \)

**Theorem 2.** \( u \in S_Q \) with \( u \geq 0 \) on \( \partial_p Q \) implies \( u \geq 0 \) on \( Q \).

**Corollary.** If \( c + \{d_j\}_j \leq 0 \) weakly on \( Q \), then the weak solution \( u \) of \( Lu = 0 \) in \( Q \), \( u = 1 \) on \( \partial_p Q \) satisfies \( 0 \leq u(x, t) \leq 1 \) on \( Q \).

From now on assume \( c + \{d_j\}_j \leq 0 \) weakly on \( Q \).

**Theorem 3.** Let \( u \in S_Q'' \). If, for some \((x_0, t_0) \in Q\), \( u(x_0, t_0) = \inf_Q u \leq 0\), then \( u(x, t) \equiv u(x_0, t_0) \) on \( \Omega \times (0, t_0) \).

**Theorem 4.** \( S_Q \subset S_Q' = S_Q'' = S_Q'' \).

**Theorem 5.** Let \( F(x) \) be convex on \( E^n \) with \( F(0) \leq 0 \). If \( Lu = 0 \) on \( Q \), then \( -F(u) \in S_Q' \).

**Theorem 6.** Let \( F(x) \) be nondecreasing and convex on \( E^n \) with \( F(0) \leq 0 \). If \( -u \in S_Q', \) then \( -F(u) \in S_Q' \).
THEOREM 7. If $u \in S'_Q$, and if $u(x, t) \geq 0$, then there exist $t_0, t_1$ with $0 \leq t_0 \leq t_1 \leq T$ such that
\[
\begin{align*}
u(x, t) &\equiv 0 & \text{on } \Omega \times (0, t_0), \\
0 < u(x, t) &< +\infty & \text{on } \Omega \times (t_0, t_1), \\
u(x, t) &\equiv +\infty & \text{on } \Omega \times (t_1, T).
\end{align*}
\]

THEOREM 8. If $u, v \in S'_Q$ and $c > 0$, then (i) $cu \in S'_Q$, (ii) $u + v \in S'_Q$, (iii) $\inf(u, v) \in S'_Q$.

THEOREM 9. $u, -u \in S'_Q$ implies $Lu = 0$ weakly on $Q$.

THEOREM 10. Let $u \in S'_Q$ and let $R$ be a standard rectangle with $\overline{R} \subset Q$. Set
\[
v(x, t) = L(u; (x, t), R) \quad (x, t) \in R, \\
v(x, t) = u(x, t) \quad (x, t) \in Q - R.
\]
Then $u \geq v$ on $Q$, $Lv = 0$ on $R$, and $v \in S'_Q$.

REFERENCES


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