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MINIMAL TOTAL ABSOLUTE CURVATURE FOR ORIENTABLE SURFACES WITH BOUNDARY

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Let M be an orientable surface with single smooth boundary curve C which is C^2 imbedded in Euclidean three-space E^3. (M may be thought of as a closed orientable surface with a single disc removed.) Let M_ε be the set of points of E^3 at a distance ε from M. M_ε is, of course, for small ε, an imbedded closed surface which is almost everywhere C^2. Using N. Grossman's [**1**] adaptation of N. Kuiper's [**2**] definition, we say that M has minimal total absolute curvature if M_ε is tightly imbedded or has the two piece property, TPP [**2**].

We announce the following result:

THEOREM. *Let M be an orientable surface of genus g with a single smooth boundary curve which is C^2 imbedded in E^3. Then M has minimal total absolute curvature if and only if M has $g=0$ and is a planar disc bounded by a convex curve.*

The proof uses a series of integral equations and geometric arguments. The outline is as follows. First, in his paper [**1**], N. Grossman shows that an orientable surface M of genus g with boundary curve C has minimal total absolute curvature only if the following integral equality holds:

$$(1) \qquad \frac{1}{2\pi}\int_M |K|\, dA + \frac{1}{2\pi}\int_C \kappa\, ds = 1 + 2g,$$

where K is the Gauss curvature of M and κ is the Frenet curvature of the boundary curve C considered as a space curve in E^3, where dA is the area element of M and ds is the arc element of C. Note that the right-hand side is the sum of the betti-numbers of M and compare with Kuiper [**2**] for closed surfaces.

Next, the theorem of Gauss-Bonnet yields

$$(2) \qquad \frac{1}{2\pi}\int_M K\, dA + \frac{1}{2\pi}\int_C \kappa_g\, ds = 1 - 2g,$$

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where κ_g is the geodesic curvature of C considered as a curve on the surface M.

Adding (1) and (2), we obtain that if M has minimal total absolute curvature,

$$\frac{2}{2\pi}\int_{M:\{K>0\}} K\,dA + \frac{1}{2\pi}\int_C (\kappa + \kappa_g)\,ds = 2, \tag{3}$$

where the first integral is taken over the points of M where $K>0$.

LEMMA 1. *If M has minimal total absolute curvature, then M has TPP.*

In [3], L. Rodriguez shows that, if M has TPP,

$$\frac{1}{2\pi}\int_{M:\{K>0\}} K\,dA + \frac{1}{2\pi}\int_C (\kappa + \kappa_g)\,ds = 2. \tag{4}$$

Subtracting (4) from (3), we obtain $(1/2\pi)\int_{M:\{K>0\}} K\,dA=0$, and hence $K\leqq 0$ in the interior of M.

LEMMA 2. *$K\leqq 0$ in the interior of M.*

LEMMA 3. *C is a plane convex curve.*

Lemma 3 is proved by using Morse theory and studying the convex hull of M_ε.

LEMMA 4. *$K\equiv 0$ in the interior of M.*

This follows immediately from Lemmas 2 and 3.

Now Lemma 4 implies $\int_M |K|\,dA=0$, and Lemma 3 implies $(1/2\pi)\int_C \kappa\,ds=1$. Thus, in order for equation (1) to hold g must be zero and M must be a planar disc bounded by a convex curve.

REFERENCES

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