MINIMAL TOTAL ABSOLUTE CURVATURE FOR ORIENTABLE SURFACES WITH BOUNDARY

BY JAMES H. WHITE

Communicated by S. S. Chern, September 22, 1973

Let $M$ be an orientable surface with single smooth boundary curve $C$ which is $C^2$ imbedded in Euclidean three-space $E^3$. ($M$ may be thought of as a closed orientable surface with a single disc removed.) Let $M_\varepsilon$ be the set of points of $E^3$ at a distance $\varepsilon$ from $M$. $M_\varepsilon$ is, of course, for small $\varepsilon$, an imbedded closed surface which is almost everywhere $C^2$. Using N. Grossman’s [1] adaptation of N. Kuiper’s [2] definition, we say that $M$ has minimal total absolute curvature if $M_\varepsilon$ is tightly imbedded or has the two piece property, TPP [2].

We announce the following result:

**Theorem.** Let $M$ be an orientable surface of genus $g$ with a single smooth boundary curve which is $C^2$ imbedded in $E^3$. Then $M$ has minimal total absolute curvature if and only if $M$ has $g=0$ and is a planar disc bounded by a convex curve.

The proof uses a series of integral equations and geometric arguments. The outline is as follows. First, in his paper [1], N. Grossman shows that an orientable surface $M$ of genus $g$ with boundary curve $C$ has minimal total absolute curvature only if the following integral equality holds:

\[
\frac{1}{2\pi} \int_M |K| \, dA + \frac{1}{2\pi} \int_C \kappa \, ds = 1 + 2g,
\]

where $K$ is the Gauss curvature of $M$ and $\kappa$ is the Frenet curvature of the boundary curve $C$ considered as a space curve in $E^3$, where $dA$ is the area element of $M$ and $ds$ is the arc element of $C$. Note that the right-hand side is the sum of the betti-numbers of $M$ and compare with Kuiper [2] for closed surfaces.

Next, the theorem of Gauss-Bonnet yields

\[
\frac{1}{2\pi} \int_M K \, dA + \frac{1}{2\pi} \int_C \kappa \, ds = 1 - 2g,
\]


Copyright © American Mathematical Society 1974
where $\kappa_g$ is the geodesic curvature of $C$ considered as a curve on the surface $M$.

Adding (1) and (2), we obtain that if $M$ has minimal total absolute curvature, 
\begin{equation}
\frac{2}{2\pi} \int_{M^i(K>0)} K\, dA + \frac{1}{2\pi} \int_C (\kappa + \kappa_g) \, ds = 2,
\end{equation}
where the first integral is taken over the points of $M$ where $K>0$.

**Lemma 1.** If $M$ has minimal total absolute curvature, then $M$ has TPP.

In [3], L. Rodriguez shows that, if $M$ has TPP, 
\begin{equation}
\frac{1}{2\pi} \int_{M^i(K>0)} K\, dA + \frac{1}{2\pi} \int_C (\kappa + \kappa_g) \, ds = 2.
\end{equation}
Subtracting (4) from (3), we obtain $(1/2\pi) \int_{M^i(K>0)} K\, dA=0$, and hence $K\leq 0$ in the interior of $M$.

**Lemma 2.** $K\leq 0$ in the interior of $M$.

**Lemma 3.** $C$ is a plane convex curve.

Lemma 3 is proved by using Morse theory and studying the convex hull of $M$.

**Lemma 4.** $K\equiv 0$ in the interior of $M$.

This follows immediately from Lemmas 2 and 3.

Now Lemma 4 implies $\int_M |K| \, dA=0$, and Lemma 3 implies $(1/2\pi) \int_C \kappa \, ds=1$. Thus, in order for equation (1) to hold $g$ must be zero and $M$ must be a planar disc bounded by a convex curve.

**References**

3. L. Rodriguez, The two-piece-property and relative tightness for surfaces with boundary (xeroxed thesis), Brown University, Providence, R.I.

Department of Mathematics, University of California, Los Angeles, California 90024