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THE MORSE-PALAIS LEMMA ON BANACH SPACES

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1. **Introduction.** In [3], Tromba gave a definition of nondegeneracy and generalized the well-known Morse-Palais lemma (see [1], [2]) to a real Banach space which is a dual space. In [4], he gave a stronger definition of nondegeneracy and proved the corresponding theorem for "any" Banach space. However, his definition of nondegeneracy in [4] implies that if the Banach space under consideration is separable then its dual space is also separable (see line 4, p. 86). Consequently, his results are not applicable to the Banach space of continuous functions on [0, 1] with sup norm. In this note we define nondegeneracy (and strong nondegeneracy) in a simpler way and prove the Morse-Palais lemma actually for any Banach space. An example is also given. We point out that our conditions for nondegeneracy are so weak that our theorem is a generalization of Palais' theorem even if the Banach space is a Hilbert space. (We do not require the invertibility of $D^2f(p)$ at a critical point p.) We remark that Uhlenbeck's definition of weak nondegeneracy [5] has no apparent relation with ours.

2. **Definitions and theorem.** Let f be a C^k function ($k \geqq 1$) defined on an open set U in a (real) Banach space B. $p \in U$ is a critical point of f if $Df(p)=0$. We will always regard $D^2f(x)$ and $D^3f(x)$ as elements of $L(B, B^*)$ and $L(B, L(B, B^*))$, respectively.

DEFINITION. Let f be at least C^2. The critical point p is *nondegenerate* if $D^2f(p)$ is injective and there exists a neighborhood $W \subset U$ of p such that (1) $D^2f(x)(B) \subset D^2f(p)(B)$ for all $x \in W$, (2) $D^2f(p)^{-1} \circ D^2f(x) \in L(B, B)$, and (3) the map $x \mapsto D^2f(p)^{-1} \circ D^2f(x)$ is continuous from W into $L(B, B)$ (operator norm topology for $L(B, B)$).

REMARK. In general, $D^2f(p)^{-1}$ is not a bounded operator in any reasonable sense. What we require in (2) is that the well-defined map $D^2f(p)^{-1} \circ D^2f(x)$ is a bounded operator of B. Note also that when B is a Hilbert space the invertibility of $D^2f(p)$ implies nondegeneracy (also the following strong nondegeneracy if f is C^3).

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DEFINITION. Let f be at least C^3. The critical point p is *strongly nondegenerate* if $D^2f(p)$ is injective and there exists a neighborhood $W\subset U$ of p such that (1) $D^3f(x)(u)(B)\subset D^2f(p)(B)$ for all $x\in W$ and $u\in B$, (2) $D^2f(p)^{-1}\circ D^3f(x)(u)\in L(B, B)$ and (3) the map $(x, u)\mapsto D^2f(p)^{-1}\circ D^3f(x)(u)$ is continuous from $W\times B$ into $L(B, B)$.

REMARK. Strong nondegeneracy implies nondegeneracy.

We may assume that W is an open ball around p. For $x\in W$ define $\theta(x)\in L(B, B^*)$ as follows: $u, v\in B$,

$$(\theta(x)u, v) = \int_0^1 (1-t)(D^2f(p+t(x-p))u, v)\,dt.$$

Note that $\theta(p)=\frac{1}{2}D^2f(p)$.

LEMMA. *Let p be nondegenerate, then* (1) $\theta(x)(B)\subset\theta(p)(B)$, $x\in W$, (2) $\xi(x)\equiv\theta(p)^{-1}\theta(x)\in L(B, B)$, $x\in W$, (3) $\xi: W\to L(B, B)$ *is continuous*, (4) $\theta(x)=\theta(p)\xi(x)$, $x\in W$, (5) $\xi(x)^*\in L(B^*, B^*)$, *when restricted to* $\theta(p)(B)$, *acts like* $\theta(x)\theta(p)^{-1}$, (6) $\theta(x)=\xi(x)^*\theta(p)$, $x\in W$. *Furthermore, if p is strongly nondegenerate then* (7) $\xi: W\to L(B, B)$ *is* C^1.

THEOREM (MORSE-PALAIS LEMMA). *Let f be a C^3 function on an open set U of a real Banach space B. Suppose $p\in U$ is a nondegenerate critical point of f. Then there exists a local homeomorphism ϕ at p such that*

$$f(x) = f(p) + \tfrac{1}{2}(D^2f(p)\phi(x), \phi(x)).$$

Furthermore, ϕ is a diffeomorphism if p is strongly nondegenerate.

REMARK. When B is a Hilbert space, we have the same conclusion as Palais, but under weaker conditions of nondegeneracy. For example, let $f(x)=\langle Kx, x\rangle$ ($\langle\ ,\ \rangle$ is the inner product), where K is an injective self-adjoint compact operator of B. Palais' theorem is not applicable, but ours is.

PROOF. Obviously, $\xi(p)=I$. It follows from (3) of the Lemma that there exists a smaller open ball $V\subset W$ around p such that $\|\xi(x)-I\|_{B,B}<\frac{1}{2}$ for all x in V. ($\|S\|_{E,F}$ denotes the operator norm of $S\in L(E, F)$.) Define

$$C(x) = \xi(x)^{1/2} = [I + (\xi(x) - I)]^{1/2}$$

by means of the power series expansion. On the other hand,

$$\|\xi(x)^* - I\|_{B^*,B^*} = \|\xi(x) - I\|_{B,B} < \tfrac{1}{2}.$$

Therefore, $[\xi(x)^*]^{1/2}$ can also be defined by the power series expansion. It is easy to see that $C(x)^*=[\xi(x)^*]^{1/2}$. Now, (4) and (6) of the Lemma tell us that $C(x)^*\theta(p)=\theta(p)C(x): B\to B^*$. Hence,

$$C(x)^*\theta(p)C(x) = \theta(p)C(x)^2 = \theta(p)\xi(x) = \theta(x).$$

From Taylor's formula, for $x \in V$,

$$\begin{aligned} f(x) &= f(p) + (\theta(x)(x-p), x-p) \\ &= f(p) + (C(x)^*\theta(p)C(x)(x-p), x-p) \\ &= f(p) + (\theta(p)C(x)(x-p), C(x)(x-p)). \end{aligned}$$

The conclusions follow by defining $\phi(x)=C(x)(x-p)$.

3. **An example.** Let C be the Banach space of real-valued continuous functions in [0, 1] with the sup norm. Define a nonlinear transformation T by

$$T(x)(t) = x(t) + \int_0^1 k(t, s, x(s))\, ds.$$

Let $f(x)=\int_0^1 T(x)(t)^2\, dt$, $x \in C$. Then 0 is a nondegenerate critical point of f if k satisfies the following conditions (1) and (2). It is strongly nondegenerate if in addition k satisfies (3).

(1) $k(t, s, u)$, $0 \leqq t, s \leqq 1$, $-\infty < u < \infty$, is continuous in t and $k(t, s, 0)=0$ for all t and s.

(2) k is C^2 in u-variable such that k_u and k_{uu} are continuous in all variables and $k_u(t, s, 0)=0$ for all t and s.

(3) k is C^3 in u-variable such that k_{uuu} is continuous in all variables.

Bibliography

1. S. Lang, *Differentiable manifolds*, Addison-Wesley, Reading, Mass., 1972.

2. R. S. Palais, *The Morse lemma for Banach spaces*, Bull. Amer. Math. Soc. **75** (1969), 968–971. MR **40** #6593.

3. A. J. Tromba, *The Morse lemma on Banach spaces*, Proc. Amer. Math. Soc. **34** (1972), 396–402. MR **45** #4461.

4. ———, *The Morse lemma on arbitrary Banach spaces*, Bull. Amer. Math. Soc. **79** (1973), 85–86.

5. K. Uhlenbeck, *Morse theory on Banach manifolds*, Bull. Amer. Math. Soc. **76** (1970), 105–106. MR **40** #6596.

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