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ON REARRANGEMENTS OF WALSH-FOURIER SERIES AND HARDY-LITTLEWOOD TYPE MAXIMAL INEQUALITIES[1]

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ABSTRACT. In this note we study the a.e. convergence properties of certain rearrangements of the Walsh-Fourier series, and maximal functions of the Hardy-Littlewood type that arise from these rearrangements.

The rearrangements are defined as follows. Let r_n be the nth Rademacher function. For $N=1, 2, \cdots$, let σ_N be a permutation of the nonnegative integers such that $\sigma_N(j)=j$ for all $j \geqq N$. If $2^N \leqq n < 2^{N+1}$, $n=\sum_{j=0}^{N} \varepsilon_j 2^j$, where $\varepsilon_j=0$ or 1 if $0 \leqq j \leqq N-1$, and $\varepsilon_N=1$, we define

$$\phi_n = \prod_{j=0}^{N} r_{\sigma_N(j)}^{\varepsilon_j}.$$

We also define $\phi_0=1$ and $\phi_1=r_0$.

If σ_N is the identity permutation, $N=1, 2, \cdots$, we recover the Walsh system. If $\sigma_N(j)=N-j-1$, $0 \leqq j \leqq N-1$, $\{\phi_n\}$ is the Walsh-Kaczmarz system. (See [**1**], [**8**] and [**12**].) In general, the system $\{\phi_n\}$ is a rearrangement of the Walsh system within dyadic blocks of indices $2^N \leqq n < 2^{N+1}$, $N=1, 2, \cdots$.

We have the following result on the a.e. convergence of Fourier series with respect to $\{\phi_n\}$. For $f \in L^1(0, 1)$, let $S_n f=\sum_{j=0}^{n-1} \phi_j \int_0^1 f\phi_j\, dt$ denote the nth partial sum of the Fourier series of f with respect to $\{\phi_n\}$, and $Mf=\sup_n |S_n f|$.

THEOREM 1. *There are absolute constants C and C_p such that*

(a) $\|Mf\|_p \leqq C_p \|f\|_p$, $f \in L^p$, $2 \leqq p < \infty$.

(b) $m\{Mf > y\} \leqq C \exp(-Cy/\|f\|_\infty)$, $y > 0$, $f \in L^\infty$.

This implies the a.e. convergence of $S_n f$ to f for $f \in L^p$, $2 \leqq p < \infty$.

If we restrict ourselves to a subclass of rearrangements, we obtain better a.e. convergence results. We say that the permutations $\{\sigma_N\}$ satisfy the

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"block condition" if for each $N=1, 2, \cdots, 0 \leqq m \leqq N-1$, there is an integer $k_{N,m}$, with $0 \leqq k_{N,m} \leqq N-m-1$, such that

$$(1) \qquad \{\sigma_N(0), \cdots, \sigma_N(m)\} = \{k_{N,m}, k_{N,m}+1, \cdots, k_{N,m}+m\}.$$

THEOREM 2. *If $\{\sigma_N\}$ satisfies the block condition, then there are absolute constants C and C_p such that*

(a) $\|mf\|_p \leqq C_p \|f\|_p, f \in L^p, 1<p<2.$

(b) $\|Mf\|_1 \leqq C\int_0^1 |f|(\log^+|f|)^3\, dx + C, f \in L(\log^+ L)^3.$

(c) *If* $\int_0^1 |f|(\log^+|f|)^2 \log^+\log^+|f|\, dx < \infty$, *then $S_n f$ converges to f a.e.*

The absolute constants C and C_p in the above theorems are independent of the permutations $\{\sigma_N\}$.

The proofs of these theorems involve a modification of the Carleson-Hunt technique (see **[3]**, **[6]** and **[7]**), and L^p boundedness of certain maximal functions of the Hardy-Littlewood type. We will only give the proofs of the estimates of the maximal functions. Complete proofs of these theorems are contained in **[11]**. They will appear elsewhere in the Vilenkin group setting in a joint paper with J. Gosselin **[5]**.

To prove Theorem 2, we will show that the maximal operator

$$f \to f^* = \sup_{0 \leqq m < N; N} E(|f| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)})$$

is of weak type (p, p) $(p>1)$. Note that for the case where σ_N is the identity permutation, $N=1, 2, \cdots$, this operator is just the usual dyadic Hardy-Littlewood operator.

LEMMA 1. *If $\{\sigma_N\}$ satisfies the block condition, then for $1<p<\infty$,*

$$m\{f^* > y\} \leqq C_p^p y^{-p} \int_0^1 |f|^p\, dx,$$

where $y>0, f \in L^p$, and $C_p \leqq p/(p-1)$.

In view of (1), this is a corollary of

LEMMA 2. *For $1<p<\infty$,*

$$m\left\{\sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m}) > y\right\} \leqq C_p^p y^{-p} \int_0^1 |f|^p\, dx,$$

where $y>0, f \in L^p$, and $C_p \leqq p/(p-1)$.

PROOF. We observe that for any L^1 function g and integers $n, m \geqq 0$

$$\begin{aligned} E(g \mid r_n, \cdots, r_{n+m}) &= E(E(g \mid r_0, \cdots, r_{n+m}) \mid r_n, \cdots, r_{n+m}) \\ &= E(E(g \mid r_0, \cdots, r_{n+m}) \mid r_n, r_{n+1}, \cdots). \end{aligned}$$

The last inequality follows from the independence of the Borel fields $\mathscr{F}(r_0, \cdots, r_{n+m})$ and $\mathscr{F}(r_{n+m+1}, r_{n+m+2}, \cdots)$. (See, for example, [**4**, p. 285].) Therefore

$$\begin{aligned} m\left\{\sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m}) > y\right\} &\leqq m\left\{\sup_n E\left(\sup_k E(|f| \mid r_0, \cdots, r_k) \mid r_n, r_{n+1}, \cdots\right) > y\right\} \\ &\leqq y^{-p} \int_0^1 \sup_k |E(|f| \mid r_0, \cdots, r_k)|^p \, dx \\ &\leqq C_p^p y^{-p} \int_0^1 |f|^p \, dx, \end{aligned}$$

where $C_p \leqq p/(p-1)$. Here we have used Doob's inequality [**10**, p. 91]. This completes the proof of Lemma 2.

Remarks. It is interesting to note that the mapping

$$f \to \sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m})$$

is not of weak type (1, 1). This accounts for the fact that the argument we use only enables us to establish the a.e. convergence result for the rearranged series for functions in the class $L(\log^+ L)^2 \log^+ \log^+ L$, whereas, for the Walsh-Fourier series, a similar argument yields the same result for functions in the class $L(\log^+ L)\log^+ \log^+ L$. (See [**9**].)

The following is an example of K. H. Moon. We will construct a sequence of functions $\{g_k\}$, $0 \leqq g_k \in L^1$, such that

$$m\left\{\sup_{n,m} E(g_k \mid r_n, \cdots, r_{n+m}) > \frac{1}{2}\right\} \geqq \frac{1}{2}, \qquad k = 1, 2, \cdots,$$

but

$$\int_0^1 |g_k| \, dx \to 0 \quad \text{as } k \to \infty.$$

For each $k = 1, 2, \cdots, j = 0, 1, \cdots$, let

$$A_{k,j} = \{r_{kj} = r_{kj+1} = \cdots = r_{kj+k-1} = 1\}.$$

Since, for each k, $\{A_{k,j}\}_{j=0}^{\infty}$ is independent, and

$$\sum_{j=0}^{\infty} m(A_{k,j}) = \sum_{j=0}^{\infty} 2^{-k} = \infty,$$

the Borel-Cantelli Lemma implies that there exists J_k such that

$$m\left(\bigcup_{j=0}^{J_k - 1} A_{k,j}\right) \geqq \frac{1}{2}.$$

For $k=1, 2, \cdots$, define

$$g_k(x) = 2^{kJ_k} \quad \text{if } x \in (0, 2^{-k-kJ_k}),$$
$$= 0 \qquad \text{otherwise.}$$

Thus we have

$$m\left\{\sup_{m,n} E(g_k \mid r_n, \cdots, r_{n+m}) > \frac{1}{2}\right\} \geqq m\left(\bigcup_{j=0}^{J_k-1} A_{k,j}\right) \geqq \frac{1}{2},$$

but

$$\int_0^1 |g_k|\, dx = 2^{-k} \to 0 \quad \text{as } k \to \infty.$$

This shows that $f \to \sup_{n,m} E(|f| \mid r_n, \cdots, r_{n+m})$ is not of weak type (1, 1).

If we relaxed the block condition on the permutations $\{\sigma_N\}$, $f \to f^*$ would not be of weak type (p, p) for any $p \geqq 1$. We consider the operator

$$f \to \sup_{0 \leqq j < m; m} E(|f| \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_m).$$

Let

$$g_n(x) = 1 \quad \text{if } x \in (0, 2^{-n-1}),$$
$$= 0 \quad \text{otherwise.}$$

Then

$$\sup_{0 \leqq j < n} E(g_n \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_n)(x)$$
$$= \frac{1}{2} \quad \text{if } x \in (0, 2^{-n-1}) \cup \bigcup_{j=1}^{n} (2^{-j}, 2^{-j} + 2^{-n-1}),$$
$$= 0 \quad \text{otherwise.}$$

Therefore,

$$m\left\{\sup_{0 \leqq j < m} E(g_n \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_m) > \frac{1}{4}\right\} \geqq (n+1)2^{-n-1}.$$

However, $\int_0^1 |g_n|^p\, dx = 2^{-n-1}$. This verifies our statement.

To prove Theorem 1, it is sufficient to have the L^p boundedness ($p \geqq 2$) of a weaker operator

$$f \to f^{**} = \sup_{0 \leqq m < N; N} E(|f_N| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)}),$$

where $f_N = E(f \mid r_0, \cdots, r_N) - E(f \mid r_0, \cdots, r_{N-1})$. Note that $f^{**} \leqq f^*$.

LEMMA 3. *For* $2 \leqq p \leqq \infty$,

$$\|f^{**}\|_p \leqq 2\, \|f\|_p, \qquad f \in L^p.$$

PROOF. For $p=2$,

$$\int_0^1 |f^{**}|^2\,dx \leqq \sum_{N=1}^{\infty} \int_0^1 \sup_{0\leqq m<N-1} |E(|f_N| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)})|^2\,dx$$
$$\leqq 4 \sum_{N=1}^{\infty} \int_0^1 |f_N|^2\,dx = 4\int_0^1 |f|^2\,dx,$$

by Doob's inequality [**10**, p. 91]. For $p=\infty$,

$$\|f^{**}\|_\infty \leqq \|f^*\|_\infty \leqq \|f\|_\infty.$$

These norm inequalities together with the Riesz convexity theorem [**2**] imply our lemma.

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