

A CHARACTERIZATION OF THE FACTORS OF ORDINARY LINEAR DIFFERENTIAL OPERATORS

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Consider the ordinary differential operator L defined by

$$(1) \quad Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0y \quad \text{for } y \in C^n(I)$$

where $p_i \in C^i(I)$ and I is any interval of the real line.

For $1 \leq k < n$, let D_k denote the class of operators Q of type

$$Qy = y^{(k)} + q_{k-1}y^{(k-1)} + \cdots + q_0y$$

with $q_i \in C^{n-k}(I)$ for $i=0, \dots, k-1$.

By $W(y_1, \dots, y_k)$ we mean the Wronskian of the class C^{k-1} functions y_1, \dots, y_k , i.e. $W(y_1, \dots, y_k) = \det[y_j^{(i-1)}]$.

In [4] it was shown that a necessary and sufficient condition for a factorization $L=RQ$ with $R \in D_{n-k}$, $Q \in D_k$ to hold is:

There exist solutions y_1, \dots, y_k of $Ly=0$ satisfying

$$(2) \quad W(y_1, \dots, y_k) \neq 0 \quad \text{on } I.$$

The factor Q has the form:

$$(3) \quad Qy = W(y_1, \dots, y_k, y) / W(y_1, \dots, y_k) \quad \text{for all } y \in C^n.$$

Here we announce a characterization of R^* —the formal adjoint of the left factor R .

For a differential operator M denote by $N(M)$ the set of all solutions y of $My=0$.

Assume y_1, \dots, y_k are in $N(L)$ satisfying (2). Let $y_1, \dots, y_k, \dots, y_n$ be a basis of $N(L)$. Define

$$\bar{z}_i = W(y_1, \dots, \hat{y}_i, \dots, y_n) / W(y_1, \dots, y_n) \quad \text{for } i = 1, \dots, n$$

where the circumflex over y_i indicates that y_i is missing and \bar{z} denotes the conjugate of the complex number z .

THEOREM. *Suppose a factorization $L=RQ$ with Q given by (3) holds. Then R is unique and*

$$(4) \quad R^*z = W(z_{k+1}, \dots, z_n, z) / W(z_{k+1}, \dots, z_n) \quad \text{for all } z \in C^n.$$

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Furthermore, given $z \in N(L^*)$, $[z, y_i] = 0$ for all $i = 1, \dots, k$ if and only if $z \in N(R^*)$ where $[,]$ is the Lagrange bilinear form associated with L i.e.

$$[u, v] = \sum_{i=0}^n \sum_{j=0}^{i-1} (-1)^j (p_i \bar{v})^{(j)} u^{(i-1-j)}$$

for $u, v \in C^n(I)$.

COROLLARY. Suppose $n = 2k$. Then $R^* = Q$ if and only if $z_j \in N(Q)$ for all $j = k+1, \dots, n$.

The special case when L is formally selfadjoint and of order $2k$ reduces to the well-known result (see Heinz [2, Satz 3 and Zusatz p. 16], W. A. Coppel [1, Theorem 19, p. 80] and M. G. Krein [3]) that $L = Q^*Q$ with Q given by (3) if and only if there exist y_1, \dots, y_k in $N(L)$ which satisfy (2) and are pairwise conjugate, i.e. $[y_i, y_j] = 0$ for all $i, j = 1, \dots, k$.

Since much more information is available about lower order operators than higher order ones—particularly for orders 2, 3 and 4—it is expected that the factorization $L = RQ$ will be useful by reducing the study of a problem to one of lower order. For example, we consider the study of disconjugacy.

It follows directly from the Pólya factorization of disconjugate operators that L is disconjugate if both R and Q are. It is also known [1] that R is disconjugate if R^* is. By applying Pólya's condition W to R^* and Q we obtain a disconjugacy criterion for L :

Functions v_1, \dots, v_p from C^{p-1} are said to have property W (or form a Markov system in the terminology of [1]) if the p Wronskians $W(v_1, \dots, v_i)$ for $i = 1, \dots, p$ are positive.

The operator L is disconjugate if, for some k with $1 \leq k < n$, there exist $y_1, \dots, y_k \in N(L)$ such that y_1, \dots, y_k and some reordering of z_{k+1}, \dots, z_n have property W .

The proof is too long to be included here and will be published elsewhere together with some related results and applications and illustrations.

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