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ON DECOMPOSITIONS OF A MULTI-GRAPH INTO SPANNING SUBGRAPHS

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1. Let G be a *multi-graph*, i.e., a finite graph with no loops. $V(G)$ and $E(G)$ denote the *vertex-set* and *edge-set* of G, respectively. For $x \in V(G)$, $d(x, G)$ denotes the *degree* (or *valency*) of x in G and $m(x, G)$ denotes the *multiplicity* of edges at x in G, i.e., the minimum number m such that x is joined to any other vertex in G by at most m edges.

A graph H is called a *spanning subgraph* of G if $V(H)=V(G)$ and $E(H)\subseteq E(G)$. Let k be any positive integer. Let

$$\sigma : G = H_1 \cup H_2 \cup \cdots \cup H_k \tag{1.1}$$

be a *decomposition* of G into k spanning subgraphs so that (1) $H_1, H_2, \cdots, H_k$ are spanning subgraphs of G; (2) $H_1, H_2, \cdots, H_k$ are pairwise edge-disjoint; and, (3) $\bigcup_{1 \leqq \alpha \leqq k} E(H_\alpha)=E(G)$. For each $x \in V(G)$, let $\nu(x, \sigma)$ denote the number of subgraphs H_α in σ such that $d(x, H_\alpha) \geqq 1$. Evidently,

$$\nu(x, \sigma) \leqq \min\{k, d(x, G)\} \quad \text{for all } x \in V(G). \tag{1.2}$$

2. Given a multi-graph G and any positive integer k, we consider the problem of determining a decomposition σ of G into k spanning subgraphs such that $\nu(x, \sigma)$ is as large as possible for each vertex $x \in V(G)$. In particular, we have proved the following two theorems.

THEOREM 2.1. *If G is a bipartite graph, then, for every positive integer k, there exists a decomposition σ of G into k spanning subgraphs such that*

$$\nu(x, \sigma) = \min\{k, d(x, G)\} \quad \textit{for all } x \in V(G). \tag{2.1}$$

THEOREM 2.2. *If G is a multi-graph, then, for every positive integer k, there exists a decomposition σ of G into k spanning subgraphs such that*

$$\begin{aligned} \nu(x, \sigma) &\geqq \min\{k - m(x, G), d(x, G)\} \quad \textit{if } d(x, G) \leqq k \\ &\geqq \min\{k, d(x, G) - m(x, G)\} \quad \textit{if } d(x, G) \geqq k, \end{aligned} \tag{2.2}$$

for all $x \in V(G)$.

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Moreover, if $W \subseteq V(G)$ *is such that*

$$W \cap \{x \in V(G): k - m(x, G) < d(x, G) < k + m(x, G)\}$$

is independent, then σ *can be so chosen that, in addition to* (2.2), *we have*

$$\nu(x, \sigma) = \min\{k, d(x, G)\} \quad \textit{for all } x \in W.$$

3. The above theorems generalize some well-known theorems in graph theory.

Let G be a multi-graph; let H be a spanning subgraph of G. H is said to be a *matching* of G if for every vertex x, $d(x, H) \leqq 1$; H is said to be a *cover* of G if for every vertex x, $d(x, H) \geqq 1$. The *chromatic index* of G, denoted by $\chi_1(G)$, is defined to be the minimum number k such that there exists a decomposition of G into k spanning subgraphs each of which is a matching of G. The *cover index* of G, denoted by $\kappa_1(G)$ is the maximum number k such that there exists a decomposition of G into k spanning subgraphs each of which is a cover of G.

Theorems 3.1 and 3.2 below are obtained from Theorem 2.1 by taking $k = \max_{x \in V(G)} d(x, G)$ and $k = \min_{x \in V(G)} d(x, G)$, respectively.

THEOREM 3.1 [1]. *If G is a bipartite graph, then,*

$$\chi_1(G) = \max_{x \in V(G)} d(x, G).$$

THEOREM 3.2 [2]. *If G is a bipartite graph, then,*

$$\kappa_1(G) = \min_{x \in V(G)} d(x, G).$$

Similarly, Theorems 3.3 and 3.4 are seen to be special cases of Theorem 2.2.

THEOREM 3.3 [3], [4]. *If G is a multi-graph, then,*

$$\chi_1(G) \leqq \max_{x \in V(G)} \{d(x, G) + m(x, G)\}.$$

THEOREM 3.4 [5]. *If G is a multi-graph, then,*

$$\kappa_1(G) \geqq \min_{x \in V(G)} \{d(x, G) - m(x, G)\}.$$

REMARK. We have also generalized Theorem 2.1 to a theorem for balanced hypergraphs which contains as special cases some theorems due to C. Berge [6].

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