ON DECOMPOSITIONS OF A MULTI-GRAAPH 
INTO SPANNING SUBGRAPHS

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1. Let G be a multi-graph, i.e., a finite graph with no loops. V(G) and 
E(G) denote the vertex-set and edge-set of G, respectively. For x \in V(G), 
d(x, G) denotes the degree (or valency) of x in G and m(x, G) denotes the 
multiplicity of edges at x in G, i.e., the minimum number m such that x 
is joined to any other vertex in G by at most m edges.

A graph H is called a spanning subgraph of G if V(H)=V(G) and 
E(H)\subseteq E(G). Let k be any positive integer. Let 

(1.1) \[ \sigma: G = H_1 \cup H_2 \cup \cdots \cup H_k \]

be a decomposition of G into k spanning subgraphs so that (1) H_1, H_2, \cdots, 
H_k are spanning subgraphs of G; (2) H_1, H_2, \cdots, H_k are pairwise edge-
disjoint; and, (3) \[ \bigcup_{1 \leq a \leq k} E(H_a) = E(G). \]

For each x \in V(G), let \( v(x, \sigma) \) denote the number of subgraphs H_a in \( \sigma \) such that d(x, H_a)\geq 1. Evidently,

(1.2) \[ v(x, \sigma) \leq \min\{k, d(x, G)\} \quad \text{for all } x \in V(G). \]

2. Given a multi-graph G and any positive integer k, we consider the 
problem of determining a decomposition \( \sigma \) of G into k spanning subgraphs 
such that \( v(x, \sigma) \) is as large as possible for each vertex \( x \in V(G). \) In 
particular, we have proved the following two theorems.

THEOREM 2.1. If G is a bipartite graph, then, for every positive integer 
k, there exists a decomposition \( \sigma \) of G into k spanning subgraphs such that 

(2.1) \[ v(x, \sigma) = \min\{k, d(x, G)\} \quad \text{for all } x \in V(G). \]

THEOREM 2.2. If G is a multi-graph, then, for every positive integer k, 
there exists a decomposition \( \sigma \) of G into k spanning subgraphs such that 

(2.2) \[ v(x, \sigma) \geq \min\{k - m(x, G), d(x, G)\} \quad \text{if } d(x, G) \leq k \]
\[ \geq \min\{k, d(x, G) - m(x, G)\} \quad \text{if } d(x, G) \geq k, \]

for all \( x \in V(G). \)
Moreover, if \( W \subseteq V(G) \) is such that
\[
W \cap \{x \in V(G) : k - m(x, G) < d(x, G) < k + m(x, G)\}
\]
is independent, then \( \sigma \) can be so chosen that, in addition to (2.2), we have
\[
\nu(x, \sigma) = \min\{k, d(x, G)\} \quad \text{for all } x \in W.
\]

3. The above theorems generalize some well-known theorems in graph theory.

Let \( G \) be a multi-graph; let \( H \) be a spanning subgraph of \( G \). \( H \) is said to be a matching of \( G \) if for every vertex \( x \), \( d(x, H) \leq 1 \); \( H \) is said to be a cover of \( G \) if for every vertex \( x \), \( d(x, H) \geq 1 \). The chromatic index of \( G \), denoted by \( \chi_1(G) \), is defined to be the minimum number \( k \) such that there exists a decomposition of \( G \) into \( k \) spanning subgraphs each of which is a matching of \( G \). The cover index of \( G \), denoted by \( \kappa_1(G) \), is the maximum number \( k \) such that there exists a decomposition of \( G \) into \( k \) spanning subgraphs each of which is a cover of \( G \).

Theorems 3.1 and 3.2 below are obtained from Theorem 2.1 by taking \( k = \max_{x \in V(G)} d(x, G) \) and \( k = \min_{x \in V(G)} d(x, G) \), respectively.

**Theorem 3.1 [1].** If \( G \) is a bipartite graph, then,
\[
\chi_1(G) = \max_{x \in V(G)} d(x, G).
\]

**Theorem 3.2 [2].** If \( G \) is a bipartite graph, then,
\[
\kappa_1(G) = \min_{x \in V(G)} d(x, G).
\]

Similarly, Theorems 3.3 and 3.4 are seen to be special cases of Theorem 2.2.

**Theorem 3.3 [3], [4].** If \( G \) is a multi-graph, then,
\[
\chi_1(G) \leq \max_{x \in V(G)} \{d(x, G) + m(x, G)\}.
\]

**Theorem 3.4 [5].** If \( G \) is a multi-graph, then,
\[
\kappa_1(G) \geq \min_{x \in V(G)} \{d(x, G) - m(x, G)\}.
\]

**Remark.** We have also generalized Theorem 2.1 to a theorem for balanced hypergraphs which contains as special cases some theorems due to C. Berge [6].
REFERENCES


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