NORMAL FIELD EXTENSIONS $K/k$ AND $K/k$-BIALGEBRAS

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Throughout the paper $K/k$ is a field extension and $p$ is the exponent characteristic.

In this paper I introduce the notion of $K/k$-bialgebra (coalgebra over $K$ and algebra over $k$) and describe a theory of finite dimensional normal field extensions $K/k$ based on a $K$-measuring $K/k$-bialgebra $H(K/k)$ (see 1.2, 1.6 and 1.10). This approach to studying $K/k$ was inspired by my conviction that a successful theory would, in view of the Jacobson-Bourbaki correspondence theorem, result from suitably equipping the endomorphism ring $\text{End}_k K$ of $K/k$ with additional structure which would effectively reflect the multiplicative structure of $K$.

Some initial parts of the theory developed here are parallel to Moss Sweedler’s very effective theory of normal extensions based on a universal cosplit $K$-measuring $k$-bialgebra (coalgebra over $k$ and algebra over $k$) [1].

In §1 the structure of $K/k$ is related to that of $H(K/k)$. At the same time, general properties of $K/k$-bialgebras are described. In §2, $K$-measuring $k$-bialgebras and semilinear $K$-measuring $K/k$-bialgebras are related, and the structure of semilinear conormal $K$-measuring $K/k$-bialgebras is described. In §3 the structure of a finite dimensional radical extension $K/k$ and that of its $K/k$-bialgebra $H(K/k)$ are described in detail in terms of the toral $k$-subbialgebra $T$ of $H(K/k)$. As an application of the theory of toral subbialgebras, a generalization of a theorem of Jacobson on finite dimensional Lie algebras of derivations of a field $K$ is given in §4.

The material outlined in this paper is the outgrowth of preliminary research described at the 1971 Ohio State University Conference on Lie Algebras and Related Topics. A complete development of this material is given in a forthcoming book [2].

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1. **K/k-bialgebras and H(K/k).** The ring End_{k}K of k-linear endomorphisms of a field extension K/k can be regarded as a K/k-algebra in the sense of the following definition.

1.1. **Definition.** A K/k-algebra is a vector space A over K together with a mapping \( \pi: A \otimes_k A \rightarrow A \) which is K-linear, A \( \otimes_k A \) being regarded as vector space over K via the left hand factor, such that A together with \( \pi \) is a k-algebra (associative algebra with identity over k).

1.2. **Definition.** \( H(K/k) \) is the union of all coclosed subsets of End_{k}K, "coclosed" being defined as follows.

1.3. **Definition.** A subset C of End_{k}K is coclosed if for each \( x \in C \), there exist elements \( x_1, x_2, \ldots, x_n \in C \) such that \( x(ab) = \sum_i x(a)x_i(b) \) for all \( a, b \in K \).

1.4. **Proposition.** \( H(K/k) \) is a coclosed K-subspace of End_{k}K and a subalgebra of End_{k}K as k-algebra.

By the above proposition, we may regard \( H(K/k) \) as K/k-algebra.

1.5. **Theorem.** There exist K-linear mappings \( \Delta: H(K/k) \rightarrow H(K/k) \otimes_k H(K/k) \) and \( \varepsilon: H(K/k) \rightarrow K \) uniquely determined by the conditions:

1. for \( x \in H(K/k) \) and \( x_1, x_2, \ldots, x_n \in H(K/k) \), \( \Delta(x) = \sum_i x \otimes x_i \) if and only if \( x(ab) = \sum_i x(a)x_i(b) \) for all \( a, b \in K \);
2. \( \varepsilon(x) = x(1_{K}) \) for all \( x \in H(K/k) \), \( 1_{K} \) being the identity of K.

1.6. **Theorem.** \( H(K/k) \) as K/k-algebra together with the mappings \( \Delta, \varepsilon \) is a K/k-bialgebra in the sense of the following definition.

1.7. **Definition.** A K/k-bialgebra is a K/k-algebra H together with mappings \( \Delta: H \rightarrow H \otimes_k H \) and \( \varepsilon: H \rightarrow K \) such that H together with \( \Delta \) and \( \varepsilon \) is a K-coalgebra and

1. \( \Delta(1_{H}) = 1_{H} \otimes 1_{H} \);
2. \( \Delta(xy) = \sum_{i,j} x_i y \otimes x_i y_j \) whenever \( x, y \in H \), \( \Delta(x) = \sum_i x \otimes x_i \) and \( \Delta(y) = \sum_j y \otimes y_j \);
3. \( \varepsilon(1_{H}) = 1_{K} \);
4. \( \varepsilon(xy) = \varepsilon(x)\varepsilon(y) \) for all \( x, y \in H \) such that \( \varepsilon(y) \in k \).

A k-bialgebra is a k/k-bialgebra. A subbialgebra respectively bi-ideal of a K/k-bialgebra (or k-bialgebra) H is a subring and subcoalgebra D/ideal and coideal P of H.

Obviously, D and H/P are K/k-bialgebras (k-bialgebras).

1.8. **Theorem.** If the dimension K:k of K over k is finite, then \( H(K/k) = \text{End}_{k}K \).

The inclusion mapping \( i: H(K/k) \rightarrow \text{End}_{k}K \) is a measuring representation of \( H(K/k) \) on K in the following sense.
1.9. DEFINITION. A measuring representation of a \( K \)-coalgebra \( H \) on a \( K/k \)-algebra \( A \) is a \( K \)-linear mapping \( \rho: H \rightarrow \text{End}_k A \) such that \( \rho(x)(1_A) = e(x)1_A \) and \( \rho(x)(ab) = \sum \rho(\varepsilon(x))(a)\rho(x)(b) \) for \( x \in H \) and \( a, b \in A \). A measuring representation of a \( K/k \)-bialgebra (\( k \)-bialgebra) \( H \) on a \( K/k \)-algebra (\( k \)-algebra) \( A \) is a mapping \( \rho: H \rightarrow \text{End}_k A \) which is a representation of \( H \) as \( k \)-algebra and a measuring representation of \( H \) as \( K \)-coalgebra (\( k \)-coalgebra).

\( H(K/k) \) together with \( i \) is a \( K \)-measuring \( K/k \)-bialgebra in the following sense.

1.10. DEFINITION. A \( K \)-measuring \( K/k \)-bialgebra (\( K \)-bialgebra) \( H \) together with a measuring representation \( \rho \) of \( H \) on \( K \). The shorthand notation \( \rho(x)(a) = x(a) \) for \( a \in K, x \in H \) is often used for measuring bialgebras (\( H, \rho \)).

For any \( K \)-measuring \( K/k \)-bialgebra (\( K \)-bialgebra) \( (H, \rho) \), let \( K^H \) be the subfield \( \{ a \in K | \rho(x)(ab) = a\rho(x)(b) \) for all \( b \in K \) and all \( x \in H \} \) and let \( \text{Kern } H = \{ x \in H | \rho(x) = 0 \} \).

1.11. THEOREM. Let \( H \) be a \( K \)-measuring \( K/k \)-bialgebra. Then \( \text{Kern } H \) is a bi-ideal of \( H \). If \( K:K < \infty \), then \( H/\text{Kern } H \) is isomorphic as \( K/k \)-bialgebra to \( H(K/K^H) \).

The above theorem has no natural counterpart for \( K \)-measuring \( k \)-bialgebras \( H \), since \( \text{Kern } H \) is not always a bi-ideal of \( H \).

Let \( \mathcal{K} = \{ k' | k' \text{ is a subfield of } K \text{ containing } k \text{ and } K:k' < \infty \} \) and \( \mathcal{S} = \{ H | H \text{ is a subbialgebra of } H(K/k) \text{ and } H:K < \infty \} \).

1.12. THEOREM. \( \mathcal{K} \) is mapped bijectively to \( \mathcal{S} \) by the mapping \( k' \mapsto H(K/k') \).

1.13. THEOREM. For \( K:k < \infty \), \( K/k \) is normal respectively radical respectively Galois if and only if \( H(K/k) \) is conormal respectively coradical respectively co-Galois in the sense of 1.15 below.

1.14. DEFINITION. A \( K \)-coalgebra \( H \) is colocal respectively cosemisimple respectively cosplit respectively cocommutative if \( H \) has a unique minimal nonzero subcoalgebra respectively \( H \) is the sum of its minimal nonzero subcoalgebras respectively every minimal nonzero subcoalgebra of \( H \) is one dimensional respectively \( \Delta(x) = \sum x_i \otimes x_i \) if and only if \( \Delta(x) = \sum x_i \otimes x_i \) for all \( x \in H \), that is, if the dual \( K \)-algebra \( H^* \) of \( H \) is local respectively semisimple respectively split respectively commutative. (Here, \( H^* \) is semisimple if every finite dimensional homomorphic image is a direct sum of fields.)

1.15. DEFINITION. A \( K/k \)-bialgebra \( H \) is conormal if \( H \) is cosplit and cocommutative and the semigroup \( G(H) \) of grouplike elements of \( H \) is a
group. If \( H \) is conormal, \( H \) is co-Galois respectively coradical if \( H \) is co-semisimple respectively colocal, that is, if

\[
H = KG(H) (\text{K-span of } G(H))/G(H) = \{1_H\}.
\]

1.16. Theorem. A \( K/k \)-bialgebra \( H \) has a unique maximal colocal subbialgebra \( H(1_H) \).

1.17. Theorem. Let \( K/k \) be finite dimensional and normal. Then \( K = K_{\text{Gal}} K_{\text{rad}} \) (internal tensor product of \( k \)-algebras) and \( H(K/k) = H_{\text{Gal}} H_{\text{rad}} \) (internal tensor product of \( k \)-algebras) where \( K_{\text{Gal}} \) and \( K_{\text{rad}} \) are Galois and radical extensions of \( k \) respectively, \( H_{\text{Gal}} \) and \( H_{\text{rad}} \) are \( K_{\text{Gal}} \)- and \( K_{\text{rad}} \)-subbialgebras of \( H \) respectively, in the sense of 1.18 below, \( H_{\text{Gal}} \) and \( H_{\text{rad}} \) stabilize \( K_{\text{Gal}} \) and \( K_{\text{rad}} \) respectively and the mappings \( x \mapsto x|_{K_{\text{Gal}}} \) and \( y \mapsto y|_{K_{\text{rad}}} \) map \( H_{\text{Gal}} \) and \( H_{\text{rad}} \) isomorphically to \( H(K_{\text{Gal}}/k) \) and \( H(K_{\text{rad}}/k) \) respectively.

A subset \( C \) of a \( K/k \)-bialgebra \( H \) is coclosed if for each \( x \in C \), there exist \( x_1, x_2, \ldots, x_n \in C \) such that \( \Delta(x) = \sum_i x_i \otimes_K x_i \). A \( k' \)-subspace \( C \) of \( H \) is linearly disjoint to \( K \) over \( k' \) if a \( k' \)-basis for \( C \) is a \( k \)-basis for the \( K \)-span \( KC \) of \( C \), \( k' \) being a subfield of \( K \) containing \( k \).

1.18. Definition. A \( k' \)-subcoalgebra of a \( K/k \)-bialgebra \( H \) is a coclosed \( k' \)-subspace \( H' \) of \( H \) containing \( 1_H \) which is linearly disjoint to \( K \) over \( k' \) and satisfies the condition \( e(H') \subseteq k' \). A \( k' \)-subbialgebra respectively \( k \)-subbialgebra of \( H \) is a subring of \( H \) which is also a \( k' \)-subcoalgebra respectively \( k \)-subcoalgebra of \( H \).

1.19. Proposition. A \( k' \)-subcoalgebra respectively \( k'/k \)-subbialgebra respectively \( k'-k \)-subbialgebra of a \( K/k \)-bialgebra \( H \) can be regarded naturally as a \( k' \)-coalgebra respectively \( k' /k \)-bialgebra respectively \( k'-k \)-bialgebra.

1.20. Theorem. For any finite dimensional normal extension \( K/k \) and for \( H = H(K/k) \), \( H(1_H) = H(K/K_{\text{Gal}}) \) and \( KG(H) = H(K/K_{\text{rad}}) \). Moreover, \( H_{\text{rad}} \) and \( H_{\text{Gal}} \) are \( K_{\text{rad}} \)- and \( K_{\text{Gal}} \)-forms of the \( K/k \)-bialgebras \( H(1_H) \) and \( KG(H) \) respectively, in the following sense.

1.21. Definition. A \( k' \)-form/\( k \)-form of a \( K/k \)-bialgebra \( H \) is a \( k' \)-subbialgebra respectively \( k \)-subbialgebra \( H' \) of \( H \) such that \( H = KH' \) (\( K \)-span of \( H' \)).

1.22. Theorem. Let \( K/k \) be finite dimensional and normal. Then the cosplit \( k \)-forms \( H \) of \( H(K/k) \) which stabilize \( K_{\text{rad}} \) and \( K_{\text{Gal}} \) are those of the form \( H = H_{\text{rad}} (kG) \) (internal tensor product of \( k \)-bialgebras) where \( H_{\text{rad}} \) is a \( k \)-form of \( H(K/K_{\text{Gal}}) \) and \( G \) is the group of automorphisms of \( K/k \).

In particular, the problem of finding a \( k \)-form for \( H(K/k) \) for \( K/k \) finite
dimensional and normal reduces to the same problem for $K/k$ finite dimensional and radical.

2. The structure of conormal $K$-measuring $K/k$-bialgebras. Let $H_k$ be a $k$-bialgebra and $\rho$ a measuring representation of $H_k$ on a $k$-algebra $A$. Then $A \otimes_k H_k$ can be regarded as $k$-algebra with product

$$(a \otimes x)(b \otimes y) = \sum_i a_i x(b) \otimes x_i y \quad (a, b \in A, x, y \in H_k),$$

called the semidirect product (smash product) of $A$ and $H$.

2.1. Proposition. Let $(H_k, \rho_k)$ be a $K$-measuring $k$-bialgebra. Then $(K \otimes_k H_k, \text{id}_K \otimes \rho_k)$ together with the semidirect product $k$-algebra structure and obvious $K$-coalgebra structure for $K \otimes_k H_k$ is a $K$-measuring $K/k$-bialgebra which is semilinear in the sense that $x(by) = \sum_i x(b)x_i y$ for all $b \in K, x, y \in K \otimes_k H_k$.

2.2. Proposition. Let $(H, \rho)$ be a semilinear $K$-measuring $K/k$-bialgebra. Let $H_k$ be a $k$-form of $H$ and let $\rho_k = \rho|_{H_k}$. Then $(H_k, \rho_k)$ is a $K$-measuring $k$-bialgebra and $(H, \rho)$ is isomorphic to $(K \otimes_k H_k, \text{id}_K \otimes \rho_k)$.

2.3. Definition. Let $C_K$ be a $K$-coalgebra, $C_k$ a $k$-coalgebra. Then one can construct the tensor product $K$-coalgebra $C_K \otimes_k C_k$. If $H_K$ is a $K/k$-bialgebra and $H_k$ a $k$-bialgebra, the tensor product $K/k$-bialgebra $H_K \otimes_k H_k$ has the tensor product $k$-algebra and $K$-coalgebra structures.

2.4. Definition. Let $H$ be a $K/k$-bialgebra. Let $H_K$ be a $K/k$-subbialgebra of $H$ and $H_k$ a $k$-subbialgebra of $H$. Then we say that $H$ is the internal semidirect product of $H_K$ and $H_k$ or that $H = H_K H_k$ (internal semidirect product $K/k$-bialgebra) if there exists a measuring representation $\rho$ of $H_k$ on $H_K$ such that the $K$-linear mapping $H_K \otimes_k H_k \to H$ induced by the product in $H$ is an isomorphism (of $k$-algebra and $K$-coalgebras) from $H_K \otimes_k H_k$ (semidirect product $k$-algebra with respect to $\rho$ and tensor product $K$-coalgebra).

The following theorem generalizes to $K/k$-bialgebras a theorem due to Bertram Kostant [I] on $k$-bialgebras.

2.5. Theorem. Let $H$ be a conormal semilinear $K$-measuring $K/k$-bialgebra. Then $H = H(1_H) k G(H)$ (internal semidirect product $K/k$-bialgebra) where $k G(H)$ is the $k$-span of $G(H)$.

2.6. Definition. A $K$-measuring $K/k$-bialgebra $(H, \rho)$ is $G(H)$-faithful if the restriction of $\rho$ to $G(H)$ is injective.

If $K_{\text{rad}}/k$ and $K_{\text{Gal}}/k$ are finite dimensional radical and Galois extensions respectively, $H_{\text{rad}}$ is a coradical $K_{\text{rad}}$-measuring $K_{\text{rad}}/k$-bialgebra and $H_{\text{Gal}}$ is a co-Galois $G(H_{\text{Gal}})$-faithful $K_{\text{Gal}}$-measuring $K_{\text{Gal}}/k$-bialgebra, then
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$H=H_{\text{rad}} \otimes_k H_{\text{Gal}}$ can be regarded naturally as conormal $G(H)$-faithful $K$-measuring $K/k$-bialgebra where $K=K_{\text{rad}} \otimes_k K_{\text{Gal}}$.

The following theorem generalizes 1.17.

2.7. **Theorem.** The finite dimensional conormal $G(H)$-faithful semilinear measuring bialgebras $H$ are precisely the $H_{\text{rad}} \otimes_k H_{\text{Gal}}$ described above.

3. **The toral structure of a radical extension $K/k$ and its $K/k$-bialgebra $H(K/k)$.** Let $K/k$ be finite dimensional.

3.1. **Definition.** A $k$-subcoalgebra ($k$-subbialgebra) $T$ of $H(K/k)$ is diagonalizable respectively toral if $t^p \in T$ for all $t \in T$, $st=ts$ for all $s$, $t \in T$ and each element of $T$ is diagonalizable respectively semisimple as linear transformation of $K$ over $k$.

3.2. **Theorem.** There is a bijective correspondence between the diagonalizable $k$-subbialgebras of $H(K/k)$ and the decompositions $K=\sum_{i \in S} K_i$ (direct sum of $k$-subspaces) such that $\{K_i|i \in S\}$ is a group under the composition $K_iK_j=\text{span} \{xy|x \in K_i, y \in K_j\}$.

3.3. **Theorem.** $K=k(x_1)\cdots k(x_n)$ (internal tensor product of $k$-algebras where $x_i^e \in k$ $(1 \leq i \leq n)$ for some integer $e>0$ if and only if $K^T=k$ for some diagonalizable $k$-subbialgebra of $H(K/k)$.

Assume throughout the remainder of the section that $K/k$ is radical. Let $L$ be the separable closure of $k$, $R=L \otimes_k K$, $\bar{k}=L \otimes_k k$, $\bar{T}=L \otimes_k T$ for any vector space $T$ over $k$. Let the group $G$ of automorphisms of $L/k$ act on $R$, $\bar{k}$, $\bar{T}$ by $g(a \otimes b)=g(a) \otimes b$ for $g \in G$. Identify $H(K/k) \otimes_k T$ and $H(R/k)$.

3.4. **Theorem.** The set of toral $k$-subcoalgebras ($k$-subbialgebras) of $H(K/k)$ is mapped bijectively to the set of $G$-stable diagonalizable $k$-subcoalgebras ($k$-subbialgebras) of $H(R/k)$ under $T \mapsto \bar{T}$, the inverse being $\bar{T} \mapsto T^G$ (fixed points of $G$ in $T$).

3.5. **Theorem.** Let $T$ be a toral $k$-subbialgebra of $H(K/k)$. Then the centralizer $H(K/k)^T=\{x \in H(K/k)|xt=tx \text{ for all } t \in T\}$ of $T$ in $H(K/k)$ is a $K^T$-form of $H(K/k)$.

The above theorem implies that $H(K/k)^T$ is a $K^T$-measuring $K^T/k$-bialgebra with respect to the measuring representation $\rho:H(K/k)^T \mapsto \text{End}_kK^T$, $\rho$ being restriction to $K^T$.

3.6. **Theorem.** For any toral $k$-subbialgebra $T$ of $H(K/k)$, $H(K|K^T)=KT$ (K-span of $T$) and $H(K^T/k) \cong H(K/k)^T/I$ where $I$ is the bi-ideal $\{x \in H(K/k)^T:x|_KT=0\}$.

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4. Lie $p$-subcoalgebras of $H(K/k)$. Let $K/k$ be a (possibly infinite dimensional) field extension.

4.1. DEFINITION. A Lie $p$-subcoalgebra of $H(K/k)$ is a $K$-subcoalgebra $C$ of $H(K/k)$ such that $[x, y] = xy - yx$ and $x^p$ are elements of $C$ for all $x, y \in C$.

4.2. THEOREM. Let $C$ be a finite dimensional colocal $K$-subcoalgebra of $H(K/k)$. Then $K^{p^n} \subseteq K^C$ for some $n$.

4.3. THEOREM. Let $C$ be a finite dimensional Lie $p$-subcoalgebra of $H(K/k)$. Then $K:K^C < \infty$.

The above theorem is proved by induction, using a more general version of Theorem 3.5.

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