SMOOTH MAPS OF CONSTANT RANK

BY ANTHONY PHILLIPS

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1. Introduction. In this announcement the Smale-Hirsch classification of immersions ([8], [5]) is extended to maps of arbitrary constant rank, under certain conditions on the source manifold.

**Theorem 1.** If $M$ is open and has a proper Morse function with no critical points of index $>k$, then the differential map $d: \text{Hom}_k(M, W) \to \text{Lin}_k(TM, TW)$ is a weak homotopy equivalence.

(A manifold with such a Morse function will be said to have geometric dimension $\leq k$. We will write $\text{geo dim } M \leq K$.)

**Notation.** $M$ and $W$ are smooth manifolds with tangent bundles $TM$, $TW$; $\text{Hom}_k(M, W)$ is the space of smooth maps of rank $k$ from $M$ to $W$, with the $C^1$-compact-open topology; $\text{Lin}_k(TM, TW)$ is the space of continuous maps: $TM \to TW$ which are fiberwise linear maps of rank $k$, with the compact open topology; $d(f) = df$.

**Remarks.** 1. Weakening the hypotheses leads to false statements. If $M$ is not open there are counterexamples when $k=\dim W$ as in [7]. Otherwise, take $M$ to be the parallelizable manifold $S^{k+1} \times R$; then the identity map of $M$ can be covered by $H \in \text{Lin}_k(TM, TM)$ but $H$ cannot be homotopic to the differential of an $f \in \text{Hom}_k(M, M)$ since such an $f$ (by Sard’s theorem) would be null-homotopic. I owe this example to David Frank.

2. When $k=\dim M$ this gives the Smale-Hirsch theorem for open manifolds, but when $k=\dim W$ this does not give the full classification of submersions [7]. The missing cases will be considered in a future article. (ADDED IN PROOF. A necessary and sufficient condition for $H \in \text{Lin}_k(TM, TW)$ to be homotopic to the differential of some $f \in \text{Hom}_k(M, W)$ is given, for arbitrary open $M$, in M. L. Gromov, *Singular smooth maps*, Mat. Zametki 14 (1973), 509–516. It is equivalent to requiring that $H$ factor through a $k$-dimensional bundle over a $k$-dimensional complex.) Immersions and submersions are the only overlap between this theorem and Feit’s classification of $k$-mersions (maps of rank everywhere $\geq k$) [2].

3. This theorem is not a special case of Gromov’s theorem [3], since

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having rank \( k \) is not an open condition in general. We will, however, make heavy use of the ideas and results of [3] throughout this work.

4. A map of constant rank is locally a submersion followed by an immersion, i.e., a subimmersion. The ensuing local “stability” is the key to our proof. It also follows that inverse images of points under a map of constant rank foliate the source manifold. An application of Theorem 1 is then this weak form of a theorem of [4]. On an open manifold \( M \) of geometric dimension \( \leq k \), any plane field \( \sigma \) of codimension \( k \) is homotopic to an integrable field. The proof is immediate, since projection onto the complementary bundle \( \sigma^\perp \) can be considered as a bundle map of rank \( k \) from \( TM \) to the tangent bundle of the total space of \( \sigma^\perp \).

The proof of Theorem 1 has two main steps: First, the manifold \( M \) is assumed to be highly coconnected; then the general case is reduced to this special one.

2. Proof for highly coconnected manifolds. Let \( a(0)=a(1)=0, a(2)=a(3)=1, \) and \( a(x)=\frac{1}{2}(x-1) \) if \( x\geq 4 \).

THEOREM 1'. Let \( \dim M=n \). If \( \text{geo dim } M \leq \min(a(n), k) \), then \( d: \text{Hom}_k(M, W) \to \text{Lin}_k(TM, TW) \) is a weak homotopy equivalence.

The theorem proving machine ([3], [6]) reduces the proof to showing that the restriction map \( \text{Hom}_k(V, W) \to \text{Hom}_k(U, W) \) has the covering homotopy property, when \( U \subset V \subset M \) are \( n \)-dimensional submanifolds, and \( V=U \cup \text{handle of index } \lambda \leq \min(a(n), k) \). This is not true in general (see Figure 1), but, as pointed out to me by Edgar Feldman, the weak covering homotopy property [1] is sufficient (this allows a preliminary vertical homotopy; see below). Using 3.2.3 of [3] (\( r \)-microflexible implies \( r \)-flexible) we can further reduce our problem to the following lemma.
WEAK MICRO-COVERING HOMOTOPY LEMMA. Suppose we are given $U, V$ as above, a compact $P$ and continuous maps $F: P \to \text{Hom}_k(V, W)$ and $f: P \times [0, 1] \to \text{Hom}_k(U, W)$ with $f_{p, 0} = F_p|U$ for $p \in P$. Then there exist $\epsilon > 0$ and a continuous $\tilde{F}: P \times [-1, 1] \to \text{Hom}_k(V, W)$ with $F_{p, -1} = F_p$ for $p \in P$, such that $\tilde{F}_{p, t}|U = f_{p, t}$ if $t \leq 0$ and $f_{p, t}$ if $0 \leq t \leq \epsilon$, for $p \in P$.

SKETCH OF PROOF. So as not to obscure the geometry, I will take $P = \text{a point}$ and leave it out of the notation. We then consider $F \in \text{Hom}_k(V, W)$ and a homotopy $f: I \to \text{Hom}_k(U, W)$ with $f_0 = F|U$.

Let us admit that the homotopy $f_t$ is defined on a “collar neighborhood” $N$ (as in [7]) of $U$ in $V$. By Remark 4 above, there exists a disc $D^n$ about any point of $N$ such that $f_t|D^n$ is a subimmersion, for $0 \leq t \leq 1$. This suffices for the case $k = \lambda = 1$ (everything is trivial when $k = 0$): we construct an isotopy of $N - U$ in itself which deforms the identity to a map pulling each component of $N - U$ through such a $D^n$, across the foliation defined there by $F$ (see Figure 2). Then after a preliminary homotopy defined by composing $F$ with this isotopy, the stability of submersions and immersions can be used to give an initial lifting, as required. The deformation corresponding to the problem of Figure 1 might be as in Figure 3.

In general $N - U \cong S^{k-1} \times D^{n-k} \times I$. The subset corresponding to the two discs would be a tubular neighborhood of the “core” $S \cong S^{k-1} \times \{0\} \times \{\frac{1}{2}\}$. If $F|S$ is an immersion, then $F$ subimmerses a tubular neighborhood $T$ of $S$, and we proceed as before, using a preliminary isotopy which draws $N - U$ through $T$, across the foliation defined in $T$ by $F$.

It is clearly sufficient to show that $S$ is isotopic to a sphere immersed by $F$; this is proved in three steps. First, using 5.2.1 of [3] and the hypothesis $\lambda \leq k$, the inclusion $i: S \to N - U$ is homotopic to an immersion $i'$.
transverse to the foliation defined by $F$; then $F \circ i'$ is also an immersion. Next, since $\lambda \leq a(n)$ this immersion can be $C^1$-approximated by an embedding $i''$. If the approximation is good enough, $F \circ i''$ will still be an immersion. Finally we use $\lambda \leq a(n)$ to conclude that $i$ and $i''$ are isotopic.

3. Theorem 1' implies Theorem 1. Pick $l$ sufficiently large so that $\text{geo dim } M \leq a(n+l)$, and let $M' = M \times \mathbb{R}^l$. This manifold satisfies the hypothesis of Theorem 1'. Give $M$ a metric and $M'$ the product metric. Let $p: M' \to M$ be the projection and $i: M \to M'$ the inclusion as $M \times \{0\}$.

I will prove that $d$ induces a bijection of connected components, i.e. that $d_\ast: \pi_0 \text{Hom}_k(M, W) \cong \pi_0 \text{Lin}_k(TM, TW)$. Higher homotopy groups can be treated analogously.

(a) $d_\ast$ is onto. Given $H \in \text{Lin}_k(TM, TW)$, the composition $H' = H \circ dp: TM' \to TW$ is homotopic to $dF$, for some $F \in \text{Hom}_k(M', W)$, by Theorem 1'. The projection $TM' \to \text{ker } H = (\text{ker } H')^\perp |M$ is therefore homotopic to an epimorphism $TM' \to \text{ker } dF$ covering $i$. It follows (see [3, 4.4.1], and [6]) that $i$ is homotopic to a smooth map $\varphi: M \to M'$ transverse to $\text{ker } dF$, and that $H$ is homotopic to $d(F \circ \varphi)$, the differential of a map of rank $k$.

(b) $d_\ast$ is one-one. Suppose given $f, g \in \text{Hom}_k(M, W)$ and a homotopy $G_t$ in $\text{Lin}_k(TM, TW)$ joining $df$ to $dg$. Composing with $dp$ gives an arc $G_t'$ joining $d(f \circ p)$ to $d(g \circ p)$; by Theorem 1' the arc $G_t'$ is homotopic with fixed endpoints to an arc $dF_t$, with $F_0 = f \circ p$, $F_1 = g \circ p$. It follows that the arc of projections $TM \to \text{ker } G_t' = (\text{ker } G_t')^\perp |M$ is homotopic to an arc of epimorphisms $TM \to \text{ker } dF_t^\perp |M$, which we consider as an arc $H_t$ of maps $TM \to TM'$, with $H_t$ transverse to $\text{ker } F_t$. Assertion. This arc is homotopic through such arcs to the arc of the differentials of an arc $\varphi_t: M \to M'$ with $\varphi_t$ transverse to $\text{ker } F_t$. We return to this assertion in a
It is easy to check, using [3] or [6] again, that \( i \) is homotopic to \( q_0 \) through maps transverse to ker \( dF_0 \), and homotopic to \( q_1 \) through maps transverse to ker \( dF_1 \), so that a homotopy in Hom\(_k(M, W)\) between \( f \) and \( g \) may be described by

\[
f = F_0 \circ i \sim F_0 \circ q_0 \sim F_1 \circ q_1 \sim F_1 \circ i = g.
\]

The assertion is an application of [3]. Let

\[ A(M) = \{ H \in \text{Lin}(TM, TM')^I \mid H_t \text{ is transverse to ker } dF_t \text{ for } t \in I \} \]

and \( B(M) = \{ f \in \text{Hom}(M, M')^I \mid d \circ f \in A(M) \} \). Here \( \text{Hom}(M, M') \) is the space of smooth maps: \( M \to M' \) with the \( C^1 \)-compact-open topology, \( \text{Lin}(TM, TM') \) is the space of continuous, fiberwise linear maps: \( TM \to TM' \) with the compact-open topology, and \( X^Y \) is the space of continuous maps: \( Y \to X \), with the compact-open topology. It follows from [3, 2.4.1, Corollary to 3.2.3 and 3.4.1] that the “differential” \( d: B(M) \to A(M) \) is a w.h.e.

REFERENCES


DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11790