SMOOTH MAPS OF CONSTANT RANK

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1. Introduction. In this announcement the Smale-Hirsch classification of immersions ([8], [5]) is extended to maps of arbitrary constant rank, under certain conditions on the source manifold.

THEOREM 1. If M is open and has a proper Morse function with no critical points of index > k, then the differential map d: Hom_k(M, W) → Lin_k(TM, TW) is a weak homotopy equivalence.

(A manifold with such a Morse function will be said to have geometric dimension ≤ k. We will write geo dim M ≤ K.)

Notation. M and W are smooth manifolds with tangent bundles TM, TW; Hom_k(M, W) is the space of smooth maps of rank k from M to W, with the C^1-compact-open topology; Lin_k(TM, TW) is the space of continuous maps: TM → TW which are fiberwise linear maps of rank k, with the compact open topology; d(f) = df.

REMARKS. 1. Weakening the hypotheses leads to false statements. If M is not open there are counterexamples when k = dim W as in [7]. Otherwise, take M to be the parallelizable manifold S^{k+1} × R; then the identity map of M can be covered by H ∈ Lin_k(TM, TM) but H cannot be homotopic to the differential of an f ∈ Hom_k(M, M) since such an f (by Sard’s theorem) would be null-homotopic. I owe this example to David Frank.

2. When k = dim M this gives the Smale-Hirsch theorem for open manifolds, but when k = dim W this does not give the full classification of submersions [7]. The missing cases will be considered in a future article. (ADDED IN PROOF. A necessary and sufficient condition for H ∈ Lin_k(TM, TW) to be homotopic to the differential of some f ∈ Hom_k(M, W) is given, for arbitrary open M, in M. L. Gromov, Singular smooth maps, Mat. Zametki 14 (1973), 509–516. It is equivalent to requiring that H factor through a k-dimensional bundle over a k-dimensional complex.) Immersions and submersions are the only overlap between this theorem and Feit’s classification of k-mersions (maps of rank everywhere ≥ k) [2].

3. This theorem is not a special case of Gromov’s theorem [3], since

having rank $k$ is not an open condition in general. We will, however, make heavy use of the ideas and results of [3] throughout this work.

4. A map of constant rank is locally a submersion followed by an immersion, i.e., a subimmersion. The ensuing local “stability” is the key to our proof. It also follows that inverse images of points under a map of constant rank foliate the source manifold. An application of Theorem 1 is then this weak form of a theorem of [4]. On an open manifold $M$ of geometric dimension $\leq k$, any plane field $\sigma$ of codimension $k$ is homotopic to an integrable field. The proof is immediate, since projection onto the complementary bundle $\sigma^\perp$ can be considered as a bundle map of rank $k$ from $TM$ to the tangent bundle of the total space of $\sigma^\perp$.

The proof of Theorem 1 has two main steps: First, the manifold $M$ is assumed to be highly coconnected; then the general case is reduced to this special one.

2. Proof for highly coconnected manifolds. Let $a(0)=a(1)=0$, $a(2)=a(3)=1$, and $a(x)=\frac{1}{2}(x-1)$ if $x\geq 4$.

**Theorem 1'.** Let $\dim M=n$. If geometric dimension $M \leq \min(a(n), k)$, then $d: \text{Hom}_k(M, W) \to \text{Lin}_k(TM, TW)$ is a weak homotopy equivalence.

The theorem proving machine ([3], [6]) reduces the proof to showing that the restriction map $\text{Hom}_k(V, W) \to \text{Hom}_k(U, W)$ has the covering homotopy property, when $U \subset V \subset M$ are $n$-dimensional submanifolds, and $V = U \cup$ handle of index $\lambda \leq \min(a(n), k)$. This is not true in general (see Figure 1), but, as pointed out to me by Edgar Feldman, the weak covering homotopy property [1] is sufficient (this allows a preliminary vertical homotopy; see below). Using 3.2.3 of [3] ($r$-microflexible implies $r$-flexible) we can further reduce our problem to the following lemma.

![Figure 1](https://www.ams.org/journal-terms-of-use)
WEAK MICRO-COVERING HOMOTOPY LEMMA. Suppose we are given $U, V$ as above, a compact $P$ and continuous maps $F: P \to \text{Hom}_k(V, W)$ and $f: P \times [0, 1] \to \text{Hom}_k(U, W)$ with $f_{p, 0} = F_p|U$ for $p \in P$. Then there exist $\varepsilon > 0$ and a continuous $\bar{F}: P \times [-1, 1] \to \text{Hom}_k(V, W)$ with $\bar{F}_{p, -1} = F_p$ for $p \in P$, such that $\bar{F}_{p, t}|U = f_{p, t}$ if $t \leq 0$ and $f_{p, t}$ if $0 \leq t \leq \varepsilon$, for $p \in P$.

SKETCH OF PROOF. So as not to obscure the geometry, I will take $P = \text{a point}$ and leave it out of the notation. We then consider $F \in \text{Hom}_k(V, W)$ and a homotopy $f: I \to \text{Hom}_k(U, W)$ with $f_0 = F|U$.

Let us admit that the homotopy $f_t$ is defined on a "collar neighborhood" $N$ (as in [7]) of $U$ in $V$. By Remark 4 above, there exists a disc $D^n$ about any point of $N$ such that $f_t|D^n$ is a subimmersion, for $0 \leq t \leq 1$. This suffices for the case $k = \lambda = 1$ (everything is trivial when $k = 0$): we construct an isotopy of $N - U$ in itself which deforms the identity to a map pulling each component of $N - U$ through such a $D^n$, across the foliation defined there by $F$ (see Figure 2). Then after a preliminary homotopy defined by composing $F$ with this isotopy, the stability of submersions and immersions can be used to give an initial lifting, as required. The deformation corresponding to the problem of Figure 1 might be as in Figure 3.

In general $N - U \cong S^{k-1} \times D^{n-k} \times I$. The subset corresponding to the two discs would be a tubular neighborhood of the "core" $S = S^{k-1} \times \{0\} \times \{1\}$. If $F|S$ is an immersion, then $F$ subimmerses a tubular neighborhood $T$ of $S$, and we proceed as before, using a preliminary isotopy which draws $N - U$ through $T$, across the foliation defined in $T$ by $F$.

It is clearly sufficient to show that $S$ is isotopic to a sphere immersed by $F$; this is proved in three steps. First, using 5.2.1 of [3] and the hypothesis $\lambda \leq k$, the inclusion $i: S \to N - U$ is homotopic to an immersion $i'$

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$\varepsilon$ This leap of faith is not required if, for $U \subseteq M$, $\text{Hom}_k(U, W)$ is defined as $\text{inj lim} \text{Hom}_k(A, W)$, where $A$ runs through the family of open neighborhoods of $U$ in $M$, and is given the quasi-topology it inherits as $\text{inj lim}$. See [3, §2].
transverse to the foliation defined by $F$; then $F \circ i'$ is also an immersion. Next, since $\lambda \leq a(n)$ this immersion can be $C^1$-approximated by an embedding $i''$. If the approximation is good enough, $F \circ i''$ will still be an immersion. Finally we use $\lambda \leq a(n)$ to conclude that $i$ and $i''$ are isotopic.

3. Theorem 1' implies Theorem 1. Pick $l$ sufficiently large so that $\text{geo dim } M \leq a(n+l)$, and let $M' = M \times R^l$. This manifold satisfies the hypothesis of Theorem 1'. Give $M$ a metric and $M'$ the product metric. Let $p : M' \to M$ be the projection and $i : M \to M'$ the inclusion as $M \times \{0\}$.

I will prove that $d$ induces a bijection of connected components, i.e. that $d_* : \pi_0 \text{Hom}_k(M, W) \cong \pi_0 \text{Lin}_k(TM, TW)$. Higher homotopy groups can be treated analogously.

(a) $d_*$ is onto. Given $H \in \text{Lin}_k(TM, TW)$, the composition $H' = H \circ dp : TM' \to TW$ is homotopic to $dF$, for some $F \in \text{Hom}_k(M', W)$, by Theorem 1'. The projection $TM' \to \ker H' = (\ker H')^\perp |M$ is therefore homotopic to an epimorphism $TM' \to \ker dF$ covering $i$. It follows (see [3, 4.4.1], and [6]) that $i$ is homotopic to a smooth map $\varphi : M \to M'$ transverse to $\ker dF$, and that $H$ is homotopic to $d(F \circ \varphi)$, the differential of a map of rank $k$.

(b) $d_*$ is one-one. Suppose given $f, g \in \text{Hom}_k(M, W)$ and a homotopy $G_t$ in $\text{Lin}_k(TM, TW)$ joining $df$ to $dg$. Composing with $dp$ gives an arc $G_t$ joining $d(f \circ p)$ to $d(g \circ p)$; by Theorem 1' the arc $G_t$ is homotopic with fixed endpoints to an arc $dF_t$, with $F_0 = f \circ p$, $F_1 = g \circ p$. It follows that the arc of projections $TM' \to \ker G_t = (\ker G_t')^\perp |M$ is homotopic to an arc of epimorphisms $TM' \to \ker dF_t^\perp |M$, which we consider as an arc $H_t$ of maps $TM' \to TM'$, with $H_t$ transverse to $\ker F_t$. Assertion. This arc is homotopic through such arcs to the arc of the differentials of an arc $\varphi_t : M \to M'$ with $\varphi_t$ transverse to $\ker F_t$. We return to this assertion in a
moment. It is easy to check, using [3] or [6] again, that $i$ is homotopic to $q_0$ through maps transverse to $\ker dF_0$, and homotopic to $q_1$ through maps transverse to $\ker dF_1$, so that a homotopy in $\text{Hom}_k(M, W)$ between $f$ and $g$ may be described by

$$f = F_0 \circ i \sim F_0 \circ q_0 \sim F_1 \circ q_1 \sim F_1 \circ i = g.$$ 

The assertion is an application of [3]. Let

$$A(M) = \{ H \in \text{Lin}(TM, TM') \mid H_t \text{ is transverse to } \ker dF_t \text{ for } t \in I \}$$

and $B(M) = \{ f \in \text{Hom}(M, M') \mid d \circ f \in A(M) \}$. Here $\text{Hom}(M, M')$ is the space of smooth maps: $M \to M'$ with the $C^1$-compact-open topology, $\text{Lin}(TM, TM')$ is the space of continuous, fiberwise linear maps: $TM \to TM'$ with the compact-open topology, and $X^Y$ is the space of continuous maps: $Y \to X$, with the compact-open topology. It follows from [3, 2.4.1, Corollary to 3.2.3 and 3.4.1] that the “differential” $d: B(M) \to A(M)$ is a w.h.e.

REFERENCES