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THE FATOU-ZYGMUND PROPERTY FOR SIDON SETS[1]

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A subset X of a discrete abelian group G is said to be a Sidon set if every bounded complex-valued function on X is the restriction to X of a Fourier-Stieltjes transform on G. In this article we give an affirmative answer to a question of J.-E. Björk [1] and N. Th. Varopoulos [6].

THEOREM 1. *Let X be a symmetric Sidon subset of G not containing 0_G. Then every bounded hermitian function on X is the restriction to X of a positive-definite function on G.*

In the terminology of Edwards, Hewitt and Ross [2], the set X has the Fatou-Zygmund property. We refer the reader to this article and to Ross [7] for a deeper understanding of the content of Theorem 1. The proof of Theorem 1 uses the technique of [3] but the presentation we give is akin to that of [4]. Unexplained notations and definitions may be found in [5].

For technical reasons we should like X to be a finite set. Thus we shall actually prove the following result.

THEOREM 2. *For all α $(0<\alpha\leqq 1)$ there is a constant $C(\alpha)$ such that for every finite symmetric Sidon (α) subset X of G not containing 0_G and every hermitian function ϕ on X with $\|\phi\|_\infty \leqq 1$, there exists μ a positive measure on $\hat{G}$ with $\|\mu\|_M \leqq C(\alpha)$ such that $\hat{\mu}|_X = \phi$.*

It is an easy consequence of Theorem 2 that the analogous statement with the word finite deleted holds. Thus Theorem 1 follows from Theorem 2. From now on let X be as in Theorem 2.

We fix n to be an even integer greater than or equal to four and define Ω to be the finite group of hermitian mappings from X to the complex nth roots of unity under pointwise multiplication. If U denotes the set of

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all hermitian functions of X into the closed unit disc we have

$$(*) \qquad U \subseteq \sec(\pi/n)\cdot \mathrm{co}(\Omega)$$

where $\mathrm{co}(\Omega)$ denotes the real-affine convex hull of Ω. This is not true if $n=2$ or if n is odd and X contains elements of order two.

The next lemma is a modification of the convolution device lemma of **[4]**.

LEMMA 3. *There exist functions g, g^*, g^+ and g^- on $G\times\Omega$ having the following properties*

(1) $g=g^+-g^-$, $g^*=g^++g^-$,

(2) $g^\pm_\omega$ *is positive definite on* G $\forall\omega\in\Omega$,

(3) $g(x,\omega)=\omega(x)$ $\forall\omega\in\Omega$, $\forall x\in X$,

(4) $\|g^\pm_\omega\|_{B(G)}\leqq\alpha^{-2}$ $\forall\omega\in\Omega$,

(5) $\|g^*_x\|_{A(\Omega)}\leqq\alpha^{-2}$ $\forall x\in G$.

PROOF. Since X is Sidon (α) there exist functions f_ω $(\omega\in\Omega)$ on G such that $f_\omega(x)=\omega(x)$ $\forall\omega\in\Omega$, $\forall x\in X$; $\|f_\omega\|_{B(G)}\leqq\alpha^{-1}$ $\forall\omega\in\Omega$. We may assume that each f_ω is hermitian on G for if not it suffices to throw away its skew-hermitian part. Thus we may write $f_\omega=f^+_\omega-f^-_\omega$ where $f^\pm_\omega$ is positive definite on G. Now define

$$g^{\pm\pm}(x,\omega)=\int f^\pm(x,\omega\lambda^{-1})f^\pm(x,\lambda)\,d\eta(\lambda)$$

where η is the invariant probability measure on Ω. We set $g^+=g^{++}+g^{--}$, $g^-=g^{+-}+g^{-+}$, $g=g^+-g^-$ and $g^*=g^++g^-$. Conditions (1)–(3) are easily checked and (4)–(5) follow as in **[4]**.

Let H denote the dual group of Ω, that is, the $\boldsymbol{Z}(n)$-module generated by X and the relations $x+(-x)=0$ $(x\in X)$. The negation mapping on X induces inversion on Ω

$$\omega(-x)=\overline{\omega(x)}=\omega^{-1}(x)$$

which in turn induces negation on H. The natural injection j of X into H given by $\langle j(x),\omega\rangle=\omega(x)$ thus satisfies $j(-x)=-j(x)$. A finite subset Y of a discrete abelian group F is said to be symmetric n-independent if and only if

(a) Y is symmetric.

(b) If $m\colon Y\to\boldsymbol{Z}$ and $\sum_{y\in Y} m(y)\cdot y=0_F$ then $m(y)-m(-y)\equiv 0 \bmod n$ for all $y\in Y$ and $m(y)\equiv 0 \bmod 2$ for all $y\in Y$ with $2y=0_F$. It is easy to prove that the subsets $j(X)$ and $\mathrm{graph}(j)=\{(x,j(x));\,x\in X\}$ are symmetric n-independent in H and $G\times H$ respectively.

LEMMA 4. *Let $0<\varepsilon\leqq 1$ and suppose that Y is a symmetric n-independent subset of F. There exist functions p^+, p^-, p^e and p^o on F such that*

(1′) $p^+=p^e+p^o$, $p^-=p^e-p^o$;

(2′) *$p^\pm$ is positive definite on F;*

(3′) $p^o(y)=1/2\varepsilon\ \forall y\in Y$;

(4′) $\|p^\pm\|_{B(F)}=1$;

(5′) $|p^e(y)|\leqq\varepsilon^2\ \forall y\in F\backslash\{0_F\}$.

The letters e and o stand for even and odd.

PROOF. Let Q denote the quotient of Y induced by the equivalence relation $y_1\sim y_2$ if and only if either $y_1=y_2$ or $y_1=-y_2$. For $q\in Q$ and $\chi\in\hat{F}$ we define

$$a_q^\pm(\chi)=1\pm\frac{\varepsilon}{2}\sum_{y\in q}\chi(y)$$

and the cosine Riesz products $p^\pm$ are defined by

$$(p^\pm)^\wedge(\chi)=\prod_{q\in Q}a_q^\pm(\chi).$$

The definition of p^e and p^o is given by (1′). The verification of (2′), (3′) and (4′) is routine—see for example [**5**, p. 124]. To prove (5′) we establish by direct calculation that

$$p^e(z)=\sum(\tfrac{1}{2}\varepsilon)^{\operatorname{card}(R)}C_R(z)$$

where the summation is over all even subsets R of Q and $C_R(z)$ is the number of partial section maps $y:R\rightarrow Y$ for which $z=\sum_{q\in R}y(q)$. The definition of symmetric n-independence ensures that for each fixed z, $C_R(z)$ is nonzero for at most one value of R. Thus

$$|p^e(z)|\leqq\sup(\tfrac{1}{2}\varepsilon)^{\operatorname{card}(R)}C_R(z).$$

Since $\operatorname{card}(q)\leqq 2$ for all q in Q it follows that $C_R(z)\leqq 2^{\operatorname{card}(R)}$. Clearly $C_\varnothing(z)=0$ for $z\neq 0_F$. Recalling that the supremum is only over sets of even cardinality we have (5′).

PROOF OF THEOREM 2. We use the notation of Lemmas 3 and 4 where $Y=\operatorname{graph}(j)$ and $F=G\times H$. We define

$$s(x,\omega)=\int[(p^+)^\wedge(x,\omega\lambda^{-1})g^+(x,\lambda)+(p^-)^\wedge(x,\omega\lambda^{-1})g^-(x,\lambda)]\,d\eta(\lambda)$$

where $\wedge$ denotes the Fourier transform in the Ω, H duality only. By (2) and (2′), s_ω is positive definite in G for each ω in Ω. By (4) and (4′),

$\|s_\omega\|_{B(G)} \leqq 2\alpha^{-2}\ \forall \omega \in \Omega$. Now we rewrite s.

$$s(x,\omega) = \int (p^o)^\wedge(x,\omega\lambda^{-1}) g(x,\lambda)\, d\eta(\lambda) + \int (p^e)^\wedge(x,\omega\lambda^{-1}) g^*(x,\lambda)\, d\eta(\lambda)$$
$$= s^o(x,\omega) + s^e(x,\omega).$$

By (3) and (3′), $s^o(x,\omega) = \frac{1}{2}\varepsilon\omega(x)\ \forall\omega \in \Omega, \forall x \in X$. By (5), (5′) and since $0_G \notin X$, $|s^e(x,\omega)| \leqq \varepsilon^2\alpha^{-2}\ \forall \omega \in \Omega, \forall x \in X$. Hence

$$|s(x,\omega) - \tfrac{1}{2}\varepsilon\omega(x)| \leqq \varepsilon^2\alpha^{-2} \qquad \forall\omega \in \Omega, \forall x \in X.$$

Now by real-affine convexity and the condition (*) we have that for each element ϕ of U there exists a positive measure μ on $\hat{G}$ such that

$$\|\mu\|_M \leqq 4\varepsilon^{-1}\alpha^{-2}\sec(\pi/n),$$
$$\|\hat{\mu}|_X - \phi\|_\infty \leqq 2\varepsilon\alpha^{-2}\sec(\pi/n).$$

Now select $\varepsilon = \frac{1}{4}\alpha^2\cos(\pi/n)$. Since $\hat{\mu}|_X - \phi$ is again hermitian on X, Theorem 2 follows by iteration. The constant $C(\alpha)$ may be taken to be $32\alpha^{-4}$.

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