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ALGEBRAIC GROUPS WITH SQUARE-INTEGRABLE REPRESENTATIONS

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1. **Introduction.** Let G be a unimodular locally compact group, Z a closed subgroup contained in the center of G such that $\text{Center}(G)/Z$ is compact. An irreducible unitary representation π of G is said to be *square-integrable* mod Z (or a member of the *discrete series* of G) if there exist vectors ϕ and ψ in the representation space of π such that:

$$\int_{G/Z} |(\pi(g)\phi:\psi)|^2 \, d\bar{g} < \infty,$$

where $(.:.)$ is the scalar product of the representation space and $d\bar{g}$ the Haar measure of G/Z.

The problem of determining the existence of the discrete series for semisimple Lie groups was solved by Harish-Chandra in [**2**].

Our aim is to study the problem for another class of nonsemisimple Lie groups. The method is to apply Mackey's machinery on group extensions [**5**] to carry out an induction argument on dimension G. We must therefore assume that G contains a closed normal subgroup H such that the representation π under consideration is determined by a single quasi-orbit on the dual $\hat{H}$ and the stability subgroup G_0 of G corresponding to this quasi-orbit must fall into the same class as that of G. The class of connected simply connected nilpotent Lie groups clearly satisfies the two above conditions and the method works for those groups. Actually the problem of determining nilpotent Lie groups having discrete series has been solved by J. Wolf and C. C. Moore by a different method in [**6**].

In this article we shall investigate the problem for a larger class of Lie groups called the U-groups.

2. **Statement of the main results.** Recall that a *U-group* was defined in [**1**] as an affine algebraic group defined over $\boldsymbol{C}$ whose radical is unipotent. Accordingly a real Lie group will be called a U-group if it is locally isomorphic to the points over $\boldsymbol{R}$ of an affine algebraic U-group defined over $\boldsymbol{R}$.

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Let G be a U-group. G is said to be *acceptable* if there is an analytic homomorphism from G into a simply connected complex algebraic U-group G_c defined over $\boldsymbol{R}$ such that $j(G)$ is the connected component of the identity in the points over $\boldsymbol{R}$ of G_c and such that the order of $\ker(j)$ is equal to a *nonzero even integer* if the fundamental group of $j(G)$ is infinite.

It is clear that if G is acceptable, then it is the semidirect product of the radical N and a maximal connected semisimple subgroup S such that j becomes an isomorphism between N and the (unipotent) radical of $j(G)$, and $\ker(j)$ is a finite subgroup contained in the center of G. Moreover S is quasi-simply connected and thus acceptable in the sense of [**3**].

THEOREM 1. *Let $G=NS$ be an acceptable U-group. Then the discrete series of G exists iff the following conditions are satisfied*:

(i) *The discrete series of S exists.*

(ii) *The discrete series of N exists.*

(iii) *The center Z of N is the connected component of the identity in the center of G.*

REMARK. Theorem 1 characterizes completely the acceptable U-groups having discrete series in view of [**2**] and [**6**]. However the method of proof of Theorem 1 also provides an algorithm to determine connected simply connected nilpotent Lie groups having discrete series as successive extensions of simpler nilpotent groups, and thus an algorithm to define U-groups having discrete series. Moreover we have:

COROLLARY. *Let $G=NS$ be an acceptable U-group satisfying the conditions of Theorem 1. Then every element of the discrete series of G is induced from a representation σ of a subgroup of the form $N'S$ where N' is a closed subgroup of N normalized by S. Furthermore*:

(i) *$\sigma|S$ is equivalent to a multiple of an element of the discrete series of S.*

(ii) *Either $\sigma|N'$ is a multiple of some character of N' or $N'/\ker(\sigma|N')$ is isomorphic to a Heisenberg group, H_n, and in that case $\sigma|N'$ is the composition of the homomorphism $N'\to H_n$ with a multiple of some element of the discrete series of H_n.*

EXAMPLE. Recall that the Heisenberg group H_n is the space $\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}$ with the multiplication $(x, y, z)\cdot(x', y', z')=(x+x', y+y', z+z'+(x:y'))$, where $(.:.)$ is the natural scalar product on $\boldsymbol{R}^n$. The center of H_n is $Z=\{(0, 0, z); z\in\boldsymbol{R}\}$. For any nontrivial character χ of Z, one can construct an irreducible representation U^χ of H_n in $L^2(\boldsymbol{R}^n)$. These representations exhaust the infinite dimensional irreducible representations of H_n and are all square-integrable mod Z.

Now let $\tau=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \boldsymbol{R})$, τ acts on H_n via the action

$$\begin{aligned}(x, y, z) &\mapsto \tau(x, y, z)\\ &= (Ax + By, Cx + Dy, z - \tfrac{1}{2}(x:y) + \tfrac{1}{2}(Ax + By:Cx + Dy))\end{aligned}$$

It can be proved that $U^{\chi,\tau}(x, y, z)=U^{\chi}(\tau(x, y, z))$ is equivalent to U^{χ} (see also **[4]** for the case $n=1$). In fact, one can construct a normalized intertwining operator $T^{\chi}(\tau)$ between U^{χ} and $U^{\chi,\tau}$ such that $\tau \mapsto T^{\chi}(\tau)$ determines a unitary representation of the two-fold covering S'_n of $\mathrm{Sp}(n, \boldsymbol{R})$. It can also be shown that every irreducible representation of the semi-direct product $G_n=H_nS'_n$ whose restriction to Z is a multiple of χ can be written in the form

$$V^{\chi}((x, y, z) \cdot \tau) = \{U^{\chi}(x, y, z) \circ T^{\chi}(\tau)\} \otimes S(\tau),$$

where $S(\tau)$ is some irreducible representation of S'_n. Moreover V^{χ} is square-integrable mod Z iff S belongs to the discrete series of S'_n.

3. **Sketch of the proof of Theorem 1.** As mentioned earlier the proof of Theorem 1 is based on the application of Mackey's theory on group extensions. Thus, let G, Z as in §1, let H be a closed normal abelian subgroup of G such that H/Z is isomorphic to a vector group. Assume also that the action of G on $\hat{H}$ is countably separated, then for every irreducible representation π of G, π/H is concentrated on a transitive quasi-orbit defined by a subset of $\hat{H}$ of the form $\lambda_0+\mathcal{O}$, $\lambda_0 \in \hat{H}$, and $\mathcal{O}$ is identified with a subset of $(H/Z)^{\wedge}$. Let G_0 be the stability subgroup of G corresponding to λ_0. Then Theorem 8.1 of **[5]** shows that there exists an irreducible representation σ of G_0 such that $\sigma|H \simeq \text{mult}\ \lambda_0$ and $\mathrm{ind}_{G_0\uparrow G}\ \sigma \simeq \pi$.

If we assume also that G_0 is unimodular and use the Fourier transforms on H to express the fact that π is square-integrable mod Z, we will get:

THEOREM 2. *π is square-integrable* mod Z *iff the following conditions are satisfied*:

(i) *$\mathcal{O}$ is open in $(H/Z)^{\wedge}$.*

(ii) *The canonical bijection $G_0\backslash G \to \mathcal{O}$ carries the invariant measure on $G_0\backslash G$ into the restriction of the Haar measure of $(H/Z)^{\wedge}$.*

(iii) *σ is square-integrable* mod H.

Moreover the Haar measures of G and H can be normalized so that the formal degree of π is the same as that of σ.

Since the stabilizer corresponding to an open orbit of a rational representation of an algebraic U-group is an algebraic U-group **[1]**, and since the orbits defined by an algebraic group of linear transformations are countably separated **[4]**, we can apply Theorem 2 to a closed normal abelian subgroup H of $G=NS$ such that $Z \subset H \subset N$ and H/Z is contained in the center of N/Z. In the case H does not exist, N can be shown to be isomorphic to some Heisenberg group. In any case an induction argument on dim G can be carried out.

REMARK. If the conditions of Theorem 1 are satisfied, the open orbit obtained in the application of Theorem 2 is identified with $(H/Z)^{\wedge}$. Thus, the harmonic analysis on G is reduced to that of the subgroup $N'S$ appearing in the corollary. The precise connection between the Plancherel measures of G and $N'S$ will be investigated in another paper.

4. An example.

Let S be the group of matrices of the form

$$\begin{pmatrix} cg & 0 & 0 \\ 0 & c^{-1}g & 0 \\ 0 & 0 & g \end{pmatrix},$$

where $c \in \boldsymbol{R}^*$ and $g \in SL(3, \boldsymbol{R})$, and let G be the semidirect product of S with $\boldsymbol{R}^9$ via the natural action of S on $\boldsymbol{R}^9$. Theorem 2 can be applied to show the existence of the discrete series of G. However neither the maximal semisimple subgroup of G ($\simeq SL(3, \boldsymbol{R})$) nor the radical of G have the discrete series. This phenomenon shows that the existence of the discrete series in the case of non-U-groups depends much on the mutual action between the radical and the maximal semisimple subgroup while in the case of U-groups this mutual action seems to play a very minor role (see condition (iii) of Theorem 1).

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