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QUASICONFORMAL EXTENSION OF HOLOMORPHIC MAPPINGS OF A BALL IN C^n

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Let C^n denote the space of n complex variables $z=(z_1, \cdots, z_n)$ with Euclidean norm $\|z\|$. The open unit ball $\{z \in C^n : \|z\| < 1\}$ is denoted by B^n. We consider holomorphic functions $f(z)=(f_1(z), \cdots, f_n(z))$, $z \in B^n$, from B^n into C^n. The second derivative of such a function is a symmetric bilinear operator, $D^2f(z)(\cdot, \cdot)$ on $C^n \times C^n$, and $D^2f(z)(z, \cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $z \times C^n$, with matrix representation

$$D^2f(z)(z, \cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \, \partial z_m} z_m \right), \qquad 1 \leqq j, k \leqq n.$$

A locally biholomorphic mapping $f(z)$ from a domain $G \subset C^n$ into C^n is said to be K-quasiconformal in G if $\|Df(z)\|^n \leqq K|\det Df(z)|$, $z \in G$, where $\| \ \|$ denotes the standard operator norm $\|A\| = \sup\{\|Aw\| : \|w\| \leqq 1\}$, $A \in \mathscr{L}(C^n)$.

The purpose of this note is to announce the following n-dimensional $(n \geqq 1)$ generalizations of one-dimensional results due to J. Becker [1].

THEOREM. *Let $f(z)$ with $Df(0)=I$ be locally biholomorphic in B^n and satisfy*

$$(1) \qquad (1 - \|z\|^2) \, \|(Df(z))^{-1} D^2 f(z)(z, \cdot)\| \leqq c, \qquad z \in B^n.$$

If $c \leqq 1$ then f is univalent in B^n and

$$\|z\|/(1 + c\,\|z\|)^2 \leqq \|f(z)\| \leqq \|z\|/(1 - c\,\|z\|)^2, \qquad z \in B^n.$$

If f is K-quasiconformal in B^n and $c<1$ then f is univalent and continuous in the closed ball, $\bar{B}^n$, and f can be extended to a quasiconformal homeomorphism of R^{2n} onto R^{2n}.

For $n=1$, (1) is $|zf''(z)/f'(z)| \leqq c/(1-|z|^2)$, the local biholomorphy implies f is 1-quasiconformal in B^1, and our theorem coincides with

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Becker's results. The quasiconformal extension of f is not holomorphic on all of C^n, but viewed as a mapping of R^{2n} to R^{2n}, it is ACL, differentiable a.e., and the dilatation is uniformly bounded a.e. (cf. [**5**, p. 115]). By arguments similar to Becker's, we derive these results from our n-dimensional generalization [**3**] of Pommerenke's theory of subordination chains [**4**], and from the following lemma.

LEMMA. *Let $f(z)$ be locally biholomorphic and K-quasiconformal in B^n. If f satisfies* (1) *with $c < 2$ then*

$$\|Df(z)\| = O(1/(1 - \|z\|)^c), \qquad z \in B^n, \tag{2}$$

and f has a continuous extension to $\bar{B}^n$ that satisfies a Lipschitz condition

$$\|f(z) - f(w)\| \leqq M\,\|z - w\|^{1-c}, \qquad z, w \in \bar{B}^n. \tag{3}$$

The proof of (2) is fairly elementary, and does not require the use of subordination chains. The proof that (2) implies (3) depends upon n-dimensional versions of classical theorems of Hardy and Littlewood [**2**, pp. 361–363].

Complete proofs of our results and details of the theory of n-dimensional subordination chains will be submitted for publication elsewhere [**3**].

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