THE EXISTENCE FOR THE SOLUTION
OF THE ELLIPTIC CAUCHY PROBLEM

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Let D be a simply connected domain of the $z=x+iy$ plane, whose boundary contains a portion σ of the x-axis. Also let $A(z, \zeta)$, $B(z, \zeta)$ be holomorphic functions for z, $\zeta \in D \cup \sigma \cup \bar{D}$, where $\bar{D}=\{z \mid \bar{z} \in D\}$. The aim of this note is to announce some recent results on the global existence for the Cauchy problem of the first order linear elliptic equations (in complex normal form):

$$(1) \qquad \partial W/\partial \bar{z} = A(z, \bar{z})W + B(z, \bar{z})\overline{W}$$

where

$$\partial/\partial \bar{z} = \tfrac{1}{2}(\partial/\partial x + i\partial/\partial y) \cdot W = u + iv.$$

The notations in this note are taken from Yu [8], [9]. For general reference to the elliptic Cauchy problem, the reader is referred to the survey article by Payne [6], and to equation (1), the reader is referred to Vekua [7] and Yu [8], [9].

By using transformation

$$W(z, \bar{z}) = W_0(z, \bar{z})\exp\int_{\zeta_1}^{z} A(z, t)\, dt$$

where ζ_1 is a fixed point in $\bar{D}$, the equation (1) can be reduced to the form

$$(2) \qquad \frac{\partial W_0}{\partial \bar{z}} = c(z, \bar{z})\overline{W}_0; \quad c(z, \bar{z}) = B(z, \bar{z})\exp\left[\int_{\zeta_1}^{\bar{z}} A^*(\bar{z}, t)\, dt - \int_{\zeta_1}^{z} A(z, t)\, dt\right].$$

The following integral representation for the solution of (1) in a simply connected domain $G \subset D \cup \sigma \cup \bar{D}$ has been established by Vekua [7], and later extended to the boundary ∂G of G by Yu [9].

LEMMA 1. *Every solution $W(z)$ of (1) in G, continuous in $G \cup \partial G$, has the integral representation*

$$
\begin{aligned}
(3) \qquad W(z) = \Big\{ & \phi(z) + \int_{z_0}^{z} \Gamma_1(z, \bar{z}, t, \zeta_0)\phi(t)\, dt \\
& + \int_{\zeta_0}^{\bar{z}} \Gamma_2(z, \bar{z}, z_0, \tau)\phi^*(\tau)\, d\tau \Big\} \exp\int_{\zeta_1}^{\bar{z}} A(z, t)\, dt,
\end{aligned}
$$

where $\phi(z)$ *is a function holomorphic in* G, *and continuous in* $G \cup \partial G$. $\Gamma_1(z, \zeta, t, \tau)$, $\Gamma_2(z, \zeta, t, \tau)$ *are holomorphic for* $z, \zeta, t, \tau \in D \cup \sigma \cup \bar{D}$, *and can be constructed explicitly by Vekua* [7]. *Conversely, if* $W(z)$ *is given by* (3), *then* $W(z)$ *is a solution of* (1) *in* G, *and continuous in* $G \cup \partial G$.

We shall also make use of the following generalized Cauchy integral formula for (2).

LEMMA 2. *Let* G *be a simply connected domain with piecewise smooth boundary* ∂G *such that* $G \cup \partial G \subset D \cup \sigma \cup \bar{D}$, *then for any solution* $W_0(z)$ *of* (2) *in* G, *continuous in* $G \cup \partial G$, *we have*

$$(4) \quad W_0(z) = \frac{1}{2\pi i} \int_{\partial G} [W(t) U(t, \bar{t}, z, \bar{z}) \, dt - (\overline{W}(t))(\overline{V}(t, \bar{t}, z, \bar{z})) \, d\bar{t}]$$

where

$$U = U_1 - U_2, \qquad V = V_1 - V_2,$$
$$U_1 = 1/(t - z) - \Gamma_1(z, \bar{z}, t, \bar{t})\log[(t - z)(\bar{t} - \bar{z})] + \Sigma_1,$$
$$V_1 = -\Gamma_2^*(\bar{z}, z, t, \bar{t})\log[(t - z)(\bar{t} - \bar{z})] + \Sigma_2,$$

and $\Sigma_1(t, \eta, z, \zeta)$, $\Sigma_2(t, \eta, z, \zeta)$, $U_2(z, \eta, z, \zeta)$, $V_2(t, \eta, z, \zeta)$ *are certain holomorphic functions of* $t, \eta, z, \zeta \in D \cup \sigma \cup \bar{D}$, *given explicitly in Vekua* [7].

Since the principal part of (1) is invariant under conformal mapping, it is no loss of generality to state the Cauchy problem for equation (1) as the following: To find a solution $w(z)$ of (1) satisfying the following condition

$$(5) \qquad\qquad w(x) = \rho(x), \qquad x \in \sigma,$$

where $\rho(x)$ is continuous on σ.

The following theorem gives a method of numerical integration for an analytic data $\rho(x)$.

THEOREM 1. *Let the analytic continuation* $\rho(z)$ *of* $\rho(x)$ *be holomorphic throughout* $D \cup \sigma \cup \bar{D}$. *Then the solution of the Cauchy problem for* (1) *is analytic for* (x, y) *in* $D \cup \sigma \cup \bar{D}$, *and can be constructed by the method of successive approximation.*

SKETCH OF THE PROOF. Solve the equation

$$(6) \quad \rho(z) = \left\{ \phi(z) + \int_0^z \Gamma_1(z, z, t, 0)\phi(t) \, dt \right.$$
$$\left. + \int_0^z \Gamma_2(z, z, 0, \tau)\phi^*(\tau) \, d\tau \right\} \exp \int_{\zeta_1}^z A(z, t) \, dt$$

by the method of successive approximation for an unknown function $\phi(z)$. Hence the desired result follows from (3).

REMARK 1.   The above method gives the integral representation for the solution in a symmetric domain (or conformal symmetric domain), and information of continuous dependence. This theorem can be considered as a continuation of Hadamard [4], Garabedian [3], Henrici [5], Colton [2], and Aziz, Gilbert, Howard [1].

Theorem 2 below establishes a necessary and sufficient condition for the existence of a solution of problems (2) and (5), and also gives a method for constructing the global solution. We shall make use of the following lemma.

LEMMA 3.   *Let*

$$G(z) = \frac{1}{2\pi i} \int_\sigma \rho(t) U \, dt - (\bar{\rho}(t)) \bar{V} \, dt.$$

Then $G(z)$ is a solution of (2) in $D \cup \bar{D}$ (off σ). Furthermore, if $\rho(\tau)$ is Hölder continuous on σ, then

(7) $G^+(x) = \tfrac{1}{2}\rho(x) + G(x), \qquad x \in \sigma$

where $G^+(x)$ is the limit function of $G(z)$ for $z \in D$ (we assume D is on the left of σ), and $G(x)$ being understood as the Cauchy principal value.

PROOF.   By direct calculation (cf. Vekua [7, p. 81]).

THEOREM 2.   *Let*

(8) $F(x) = \rho(x) - G^+(x).$

Then $\rho(x) = W_0(x)$ is a Cauchy data of a solution $W_0(z)$ of (2) in D if and only if $F(x)$ is analytic and its analytic continuation $F(z)$ is a holomorphic function in $D \cup \sigma \cup \bar{D}$.

SKETCH OF THE PROOF.   Let

(9) $H(z, \bar{z}) = W_0(z) - G(z) = \frac{1}{2\pi i} \int_{\partial D \backslash \sigma} W_0 U \, dt - \overline{W_0} \bar{V} \, d\bar{t}.$

Then $W_0(z)$ is a solution in D if and only if $H(z, z) = F(z)$ is holomorphic in $D \cup \sigma \cup \bar{D}$.

REMARK 2.   By Theorem 1. We can construct a solution $H(z, \bar{z})$ of (2) in $D \cup \sigma \cup \bar{D}$ such that $H(x, x) = F(x)$. Hence $W_0(z) = H(z, \bar{z}) + G(z)$ is the solution of the Cauchy problem (2) and (5) in D.

REMARK 3.   Theorem 2 is a generalization of Hadamard [4].

REMARK 4.   Since the principal part of (1) is invariant under conformal mapping, the above theorem can be extended to a more general domain with nonanalytic initial curve.

REMARK 5.   In a forthcoming paper [10], we shall give another version of necessary and sufficient conditions for existence. We shall also study the estimate of the 'stability' and the construction for an approximate solution with imprecise data.

REMARK 6.   The results in this note can be extended to equation (1) with nonanalytic coefficients [12] and to some higher order elliptic equations [11].

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