BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

HARMONIC QUASICONFORMAL MAPPINGS OF RIEMANNIAN MANIFOLDS

BY SAMUEL I. GOLDBERG[1] AND TORU ISHIHARA

Communicated by S. S. Chern, August 1, 1973

1. **Introduction.** In this note, we announce some results concerning the distance-volume-decreasing property of harmonic quasiconformal mappings of Riemannian manifolds. Details will appear elsewhere.

Let M and N be C^∞ Riemannian manifolds of dimensions m and n, respectively. Let $f: M \to N$ be a C^∞ mapping. The Riemannian metrics of M and N can be written locally as $ds_M^2 = \omega_1^2 + \cdots + \omega_m^2$ and $ds_N^2 = \omega_1^{*2} + \cdots + \omega_n^{*2}$, where ω_i $(1 \leqq i \leqq m)$ and ω_a^* $(1 \leqq a \leqq n)$ are linear differential forms in M and N, respectively. The structure equations in M are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji},$$
$$d\omega_{ij} = \sum_j \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.$$

Similar equations are valid in N and we will denote the corresponding quantities in the same notation with asterisks. Let $f^* \omega_a^* = \sum_i A_i^a \omega_i$. Then the covariant differential of A_i^a is defined by

$$DA_i^a \equiv dA_i^a + \sum_j A_j^a \omega_{ji} + \sum_b A_i^b \omega_{ba}^* \equiv \sum_j A_{ij}^a \omega_j$$

with $A_{ij}^a = A_{ji}^a$. The mapping f is called harmonic (resp. totally geodesic) if $\sum_i A_{ii}^a = 0$ (resp. $A_{ij}^a = 0$).

If $m = n$, then at each point of M the matrix (A_i^a) has the adjoint (B_a^i). Let C be the scalar invariant $\sum B_a^i B_b^k A_{kj}^a A_{ij}^b$. In [**2**], Chern and one of the authors proved the following theorems which may be regarded as extensions of Schwarz's lemma.

THEOREM I. *Let B^n be the n-dimensional open ball with the standard hyperbolic metric and N an n-dimensional Riemannian manifold. Let $f: B^n \to N$ be a harmonic mapping satisfying the condition $C \leqq 0$. If N is an Einstein manifold with scalar curvature $R^* \leqq -4n(n-1)$ or if the sectional curvature of N is $\leqq -4$, then f is volume-decreasing.*

AMS (MOS) subject classifications (1970). Primary 53A99, 53C99, 30A60.

Key words and phrases. Harmonic mappings, quasiconformal mappings, Schwarz's lemma, distance-volume decreasing.

[1] This research was supported in part by the National Science Foundation. Science Research Council Professor at the University of Liverpool.

THEOREM II. *Let E^n be the n-dimensional euclidean space and let N be a Riemannian manifold of the same dimension. Let $f: E^n \to N$ be a harmonic mapping satisfying the condition $C \leqq 0$. If N is an Einstein manifold with negative scalar curvature which is bounded away from zero or if the sectional curvature of N is negative and bounded away from zero, then f is volume-decreasing.*

The differential f_* of f is extended to the mapping $\bigwedge^p f_*: \bigwedge^p T(M) \to \bigwedge^p T(N)$, i.e. the pth exterior power of f_*. $\bigwedge^p f_*$ is also regarded as an element of $\bigwedge^p T^*(M) \otimes \bigwedge^p T(N)$ on which a norm is defined in terms of the Riemannian metrics of M and N. The norm $\|\bigwedge^p f_*\|$ is regarded as the ratio function of intermediate volume elements of M and N [7]. In particular, $\|\bigwedge^1 f_*\| = \|f_*\|$ may be considered as the ratio of distances. When $m=n$, $\|\bigwedge^n f_*\|$ is the ratio of volume elements.

The Laplacian of $\|\bigwedge^n f_*\|$ in the case $m=n$ plays an important role in [2]. In this paper, we apply the Laplacian Δ to $\|f_*\|^2$ and obtain the following formula when f is harmonic.

$$(1) \quad (1/2)\Delta \|f_*\|^2 = \sum_{a,i,j} (A^a_{ij})^2 + \sum_{a,i,j} R_{ij} A^a_i A^a_j - \sum_{a,b,c,d;i,j} R^*_{abcd} A^a_i A^b_j A^c_i A^d_j,$$

where R_{ij} is the Ricci tensor of M (see also [3], [4]). This formula leads to several extensions of Schwarz's lemma as well as a generalization of Liouville's theorem and the little Picard theorem.

2. **Quasiconformal mappings.** At each point $x \in M$, let A be the matrix representation of $(f_*)_x$ relative to orthonormal bases of $T_x(M)$ and $T_{f(x)}(N)$ and let ${}^tA$ be the transpose of A. In the sequel, we assume rank $f_* =$ rank $A = k$ at every point. Then $k \leqq \min(m, n)$ and rank $G = k$, where G is the positive semidefinite symmetric matrix ${}^tAA$. Let $\lambda_1 \geqq \cdots \geqq \lambda_k > \lambda_{k+1} = \cdots = \lambda_m = 0$ be the eigenvalues of G. The norm $\|\bigwedge^p f_*\|$ is represented as

$$(2) \qquad \|\bigwedge^p f_*\|^2 = \sum_{i_1 < \cdots < i_p}^{k} \lambda_{i_1} \cdots \lambda_{i_p}.$$

Hence, from Newton's inequalities, we obtain

LEMMA 1. *Let $k \leqq \min(m, n)$ and suppose* rank f_* *is k everywhere on M. Then,*

$$\left(\|\bigwedge^p f_*\|^2 \Big/ \binom{k}{p} \right)^{1/p} \geqq \left(\|\bigwedge^q f_*\|^2 \Big/ \binom{k}{q} \right)^{1/q}, \qquad 1 \leqq p \leqq q \leqq k.$$

The notion of a K-quasiconformal mapping of Riemannian manifolds is now extended to manifolds of different dimensions. (This should result in an extension of Wu's work on normal families of holomorphic mappings

[8].) At each point $x \in M$, let S^{k-1} be a unit $(k-1)$-sphere in $T_x(M)$. If $(f_*)_x$ has maximal rank k, that is, if $\operatorname{rank}(f_*)_x = k = \min(m, n)$, the image of S^{k-1} under $(f_*)_x$ is an ellipsoid of dimension $k-1$.

DEFINITION. Let f be a C^∞ mapping of maximal rank k $(=\min(m, n))$ and $K \geqq 1$. f is *K-quasiconformal* if at each point x of M, the ratio of the largest to the smallest axis of the ellipsoid $(f_*)_x(S^{k-1})$ in $T_{f(x)}(N)$ $\leqq K$.

One may verify that f is K-quasiconformal if and only if $\lambda_1/\lambda_k \leqq K^2$ at each point. Hence, from (2) we obtain

LEMMA 2. *If f is K-quasiconformal, then*

$$\left(\|\Lambda^p f_*\|^2 \Big/ \binom{k}{p}\right)^{1/p} \leqq K^2 \left(\|\Lambda^q f_*\|^2 \Big/ \binom{k}{q}\right)^{1/q}, \qquad 1 \leqq p < q \leqq k.$$

3. **Statement of results.** First, with no assumption on the quasiconformality of f, formula (1) yields

PROPOSITION 1. *Let M be a compact manifold and N a manifold with nonpositive sectional curvature. Let f be a harmonic mapping of M into N. If M is an Einstein manifold with positive scalar curvature R, or if the sectional curvature of M is positive, then f is a constant mapping.*

PROPOSITION 2. *Let the sectional curvature of N be nonpositive and f be a totally geodesic mapping. If M is an Einstein manifold with positive scalar curvature R, or if the sectional curvature of M is positive, then f is a constant mapping.*

In the case when M and N have the same dimension n, then by means of Lemma 2, Theorem I gives

PROPOSITION 3. *Under the conditions in Theorem* I *with f a K-quasiconformal mapping,*

$$\|\Lambda^p f_*\|^2 \leqq \binom{n}{p} K^{2p}, \qquad 1 \leqq p \leqq n.$$

In particular,

$$f^*(ds_N^2) \leqq nK^2\, ds_B^2.$$

We return now to the case where the dimensions of M and N are m and n, respectively.

THEOREM 1. *Let M and N be Riemannian manifolds of dimensions m and n respectively. Let $f: M \to N$ be a harmonic K-quasiconformal mapping with the function $\|f_*\|$ attaining its maximum on M. If*

(a) *the sectional curvature of M is bounded below by a nonpositive constant $-A$, or M is an Einstein manifold with scalar curvature $R \geqq -m(m-1)A$,* *and*

(b) *the sectional curvature of N is bounded above by a negative constant $-B$, then*

$$\|\wedge^p f_*\|^{2/p} \leqq (m - 1/k - 1)\binom{k}{p}^{1/p}(A/B)K^4 \qquad 1 \leqq p \leqq k,$$

where $k=\min(m, n)$.

This theorem improves and generalizes the results of Kiernan [5] and one of the authors [4]. The proof proceeds by first taking a maximum point x of $\|f_*\|^2$. Then $\Delta_x\|f_*\|^2\leqq 0$ and formula (1) yield

$$2\,\|\wedge^2 f_*\|_x^2 \leqq (m-1)(A/B)\,\|f_*\|_x^2.$$

From this, together with Lemmas 1 and 2, the results follow.

In the case when $k=m=n$, Theorem 1 implies that f is volume-decreasing provided $B=AK^4$. Moreover, for the ratio $\|f_*\|$ of distances we have the following.

COROLLARY. *Under the assumptions in Theorem* 1,

$$\|f_*(X)\|^2 \leqq (m - 1/k - 1)k(A/B)K^4\,\|X\|^2$$

for every tangent vector $X \in T(M)$. If $B=(m-1/k-1)kAK^4$, then f is distance-decreasing.

If in Proposition 3 the assumption that the curvature of N is $\leqq -4$ is replaced by the assumption that the curvature of N is $\leqq -4K^4$ then the condition on the invariant C may be removed.

THEOREM 2. *Let B^m be the m-dimensional unit open ball with the hyperbolic metric of constant curvature -4. Let N be a Riemannian manifold with sectional curvature bounded above by a negative constant $-B$. Then, if $f\colon M\to N$ is a harmonic K-quasiconformal mapping*

$$\|\wedge^p f_*\|^2 \leqq \left[4\left(\frac{m-1}{k-1}\right)\frac{K^4}{B}\right]^p\binom{k}{p}, \qquad 1 \leqq p \leqq k,$$

where $k=\min(m, n)$.

COROLLARY 1. *Under the conditions in Theorem* 2 *with* $B=4k(m-1/k-1)K^4$, *the mapping f is distance-decreasing.*

COROLLARY 2. *If in addition to the hypotheses of Theorem* 2, $\dim N=m$ *and* $B=4K^4$, *the mapping f is volume-decreasing.*

REMARK. If $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_k\geqq\lambda_{k+1}\geqq\cdots\geqq\lambda_m\geqq 0$, the above results are still valid if the condition that f be K-quasiconformal is replaced by the condition (Q): At each point x of M, $\lambda_1\leqq K^2\lambda_k$ where $K\geqq 1$ is a given constant and $k=\min(m, n)$.

DEFINITION. A smooth mapping f of an m-dimensional Riemannian manifold M into an n-dimensional Riemannian manifold N satisfying the condition (Q) is called a *K-quasiconformal mapping in the generalized sense*.

Regarding Theorem II, the technique employed in establishing Theorem 2 also yields

THEOREM 3. *Let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero, and let $f: E^m \to N$ be a harmonic K-quasiconformal mapping in the generalized sense. Then, f is a constant mapping.*

The classical theorem of Liouville states that every bounded holomorphic function on the entire complex plane C is a constant. On the other hand, an entire function with two lacunary values must be a constant. This is the little Picard theorem. Theorem 3 generalizes Liouville's theorem as well as the little Picard theorem, the latter case being a consequence of the fact that the Gaussian plane minus two points carries a Kaehler metric of constant negative curvature.

BIBLIOGRAPHY

1. S. S. Chern, *On holomorphic mappings of hermitian manifolds of the same dimension*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 157–170. MR 38 #2714.

2. S. S. Chern and S. I. Goldberg, *On the volume-decreasing property of a class of real harmonic mappings*, Amer. J. Math. (to appear).

3. J. Eells Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR **29** #1603.

4. T. Ishihara, *A remark on harmonic quasiconformal mappings*, J. Math. Tokushima Univ. **5** (1971), 17–23. MR **46** #829.

5. P. J. Kiernan, *Quasiconformal mappings and Schwarz's lemma*, Trans. Amer. Math. Soc. **148** (1970), 185–197. MR **41** #467.

6. S. Kobayashi, *Volume elements, homomorphic mappings and Schwarz's lemma*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 253–260. MR **38** #6100.

7. Y. C. Lu, *Holomorphic mappings of complex manifolds*, J. Differential Geometry **2** (1968), 299–312. MR **40** #3482.

8. H. Wu, *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233. MR **37** #468.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LIVERPOOL, LIVERPOOL, ENGLAND

DEPARTMENT OF MATHEMATICS, TOKUSHIMA UNIVERSITY, JOSANJIMA, JAPAN