Let $\Omega$ be a $\sigma$-finite measure space. Let $K$ be a (nonlinear) monotone operator and let $(Fu)(x) = f(x, u(x))$ be a Nemytski operator. We consider the Hammerstein type equation

$$u + KFu = g.$$  

A detailed discussion and a complete bibliography about equation (1) can be found in [3]. The new feature about the results we present here is the fact that we do not assume any coercivity for $F$. When $F$ is monotone and $K$ maps $L^1(\Omega)$ into $L^\infty(\Omega)$, there is no growth restriction on $F$ either (cf. Theorem 1). The monotonicity of $F$ can be weakened when $K$ is compact (cf. Theorem 4). Also some of these results are valid for systems in the case where $F$ is the gradient of a convex function (cf. Theorem 5).

Assume

(2) $K$ is a monotone hemicontinuous mapping from $L^1(\Omega)$ into $L^\infty(\Omega)$ which maps bounded sets into bounded sets,

(3) $f(x, r) : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in $r$ for a.e. $x \in \Omega$, and is integrable in $x$ for all $r \in \mathbb{R}$.

**Theorem 1.** Under the assumptions (2) and (3), equation (1) has one and only one solution $u \in L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$.

**Uniqueness.** Let $u_1$ and $u_2$ be two solutions of (1). By the monotonicity of $K$ we get

$$\int_{\Omega} (u_1(x) - u_2(x)) \cdot (f(x, u_1(x)) - f(x, u_2(x))) \, dx \leq 0$$

which implies that $f(x, u_1(x)) = f(x, u_2(x))$ a.e. on $\Omega$ and therefore by (1), $u_1 = u_2$.

In proving existence of $u$ we shall use the following

**Lemma 1.** Let $X$ be a Banach space and let $K : X \to X^*$ and $F : X^* \to X$ be two monotone hemicontinuous operators. Let $\{u_n\} \subseteq X^*$, $\{v_n\} \subseteq X$ and

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{w_n} \subset X^* be three sequences such that
(4) \( u_n \) converges to \( u \) in \( X^* \) for the weak* topology,
(5) \( F(u_n) \) converges to \( v \) in \( X \) for the weak topology,
(6) \( v_n \) converges to \( v \) in \( X \) for the weak topology,
(7) \( K v_n \) converges to \( g - u \) in \( X^* \) for the weak* topology,
(8) \( \langle w_n, F(u_n) \rangle - \langle K v_n, v_n \rangle \to 0 \),
(9) \( \langle g_n, F(u_n) \rangle \to \langle g, v \rangle \) where \( g_n = u_n + w_n \).
Then \( u + K F u = g \).

**Proof of Lemma 1.** We have
\[
\langle u_n - u, F(u_n) \rangle = \langle g_n - w_n - u, F(u_n) \rangle.
\]
By the monotonicity of \( K \) we get
\[
\langle K v_n, v_n \rangle \geq \langle K v_n, v \rangle + \langle K v, v_n - v \rangle
\]
and thus
\[
\lim \inf \langle K v_n, v_n \rangle \geq \langle g - u, v \rangle.
\]
By (8) we have
\[
\lim \inf \langle w_n, F(u_n) \rangle \geq \langle g - u, v \rangle.
\]
Consequently, \( \lim \sup \langle u_n - u, F(u_n) \rangle \leq 0 \). Since \( F \) is pseudomonotone (cf. [1]), we conclude that \( v = F u \) and \( \langle u_n, F(u_n) \rangle \to \langle u, v \rangle \). Also \( \langle K v_n, v_n \rangle \to \langle g - u, v \rangle \) since \( \langle w_n, F(u_n) \rangle = \langle g_n - u_n, F(u_n) \rangle \to \langle g, v \rangle - \langle u, v \rangle \). Thus
\[
\lim \langle K v_n, v_n - v \rangle = 0,
\]
and again, since \( K \) is pseudomonotone, we conclude that \( g - u = K v = K F u \).

**Proof of Theorem 1.** By a shift we can always assume that \( f(x, 0) = 0 \) and that \( K 0 = 0 \) (note that (1) can be written as \( u + K F u = g \), where \( F = F_0 - F_0 \), \( K_0 = K(v + F(0)) - K F 0 \) and \( g = g - K F 0 \)). Let \( \Omega_n \) be an increasing sequence of finite measure subsets of \( \Omega \) such that \( \bigcup_n \Omega_n = \Omega \). Let \( \chi_n \) be the characteristic function of \( \Omega_n \). Let \( F_n \) be \( F \) truncated by \( n \), i.e.,
\[
f_n(x, r) = f(x, r) \quad \text{whenever } |f(x, r)| < n,
\]
\[
= nf(x, r)/|f(x, r)| \quad \text{whenever } |f(x, r)| \geq n.
\]
The equation
\[
(10) \quad u_n + \chi_n K \chi_n F_n(u_n) = \chi_n g
\]
has a solution.
Indeed the mapping \( K_n : v \to \chi_n K \chi_n v \) is monotone hemicontinuous from \( L^2(\Omega) \) into itself.
On the other hand, the (multivalued) operator \( A \) defined on \( L^2(\Omega) \) by
\[
A v = \{ w \in L^2(\Omega) ; v(x) = \chi_n(x) f_n(x, w(x)) \text{ a.e. on } \Omega \}
\]
is maximal monotone in $L^2(\Omega)$ and $D(A)$ is bounded in $L^2(\Omega)$ \((|v|_{L^2} \leq n(\text{ meas } Q)^{1/2}, v \in D(A))\). Consequently, $R(A+K_n)=L^2(\Omega)$ (cf. [2]) and (10) has a solution.

Multiplying (10) through by $F_n(u_n)$ and using the monotonicity of $K$ we get

\begin{equation}
\int_{\Omega} u_n \cdot F_n(u_n) \, dx \leq \int_{\Omega} \chi_n g F_n(u_n) \, dx.
\end{equation}

Let $C=2\|g\|_{L^\infty}$; we have

\[
\int_{\Omega} u_n F_n(u_n) \, dx = \int_{|u_n| \geq C} u_n F_n(u_n) \, dx + \int_{|u_n| < C} u_n F_n(u_n) \, dx \\
\leq C \int_{|u_n| \geq C} |F_n(u_n)| \, dx - C \int_{|u_n| < C} |F_n(u_n)| \, dx \\
\leq C \int_{\Omega} |F_n(u_n)| \, dx - 2C \int_{|u_n| < C} |F_n(u_n)| \, dx.
\]

Using (11) we obtain

\[
\int_{\Omega} |F_n(u_n)| \, dx \leq 4 \int_{|u_n| \leq C} |F_n(u_n)| \, dx \leq 4 \int_{|u_n| \leq C} |f(x, u_n(x))| \, dx \leq C'
\]

by assumption (3).

Going back to (10), we conclude that $\{u_n\}$ remains bounded in $L^\infty(\Omega)$. Therefore, by assumption (3), there is some function $h \in L^1(\Omega)$ such that

\begin{equation}
|F_n(u_n)(x)| \leq |f(x, u_n(x))| \leq h(x) \quad \text{a.e. on } \Omega.
\end{equation}

We apply now Lemma 1 with $v_n=\chi_n F_n(u_n), \ w_n=\chi_n K v_n, \ g_n=\chi_n g$. By extracting a subsequence, we can always assume that

- $u_n$ converges to $u$ \(u\)-weak* in $L^\infty(\Omega)$,
- $F(u_n)$ converges to $v$ weakly in $L^1(\Omega)$,
- $v_n$ converges to $v$ weakly in $L^1(\Omega)$,
- $g_n$ converges to $g$ \(u\)-weak* in $L^\infty(\Omega)$.

Hence

- $w_n$ converges to $g-\nu$ \(u\)-weak* in $L^\infty(\Omega)$,
- $K v_n$ converges to $g-\nu$ \(u\)-weak* in $L^\infty(\Omega)$.

It remains to verify (8) and (9). We have

\[
\langle w_n, F(u_n) \rangle = \int_{\Omega} \chi_n K v_n \cdot F(u_n) \, dx = \int_{\Omega} K v_n \chi_n F(u_n) \, dx \\
= \int_{\Omega} K v_n \cdot v_n \, dx + \int_{\Omega} \chi_n K v_n (F(u_n) - F_n(u_n)) \, dx.
\]
The last term can be bounded by
\[ C \int_{|F(u_n)| > n} |F u_n| \, dx \leq C \int_{|h| > n} |h(x)| \, dx \]
which tends to zero as \( n \to +\infty \) and (8) follows.

Finally (9) holds since
\[ \langle g_n, F(u_n) \rangle = \int_{\Omega} \chi_n g F(u_n) \, dx = \int_{\Omega} g F(u_n) \, dx + \int_{\Omega} (\chi_n - 1)g F(u_n) \, dx, \]
and the last term goes to zero by Lebesgue's theorem.

**Theorem 2 (Continuous Dependence).** Under the assumptions (2) and (3), \((I+KF)^{-1}\) is strongly continuous from \(L^\infty(\Omega)\) into \(L^1(\Omega)\) and \((I+KF)^{-1}\) is demicontinuous (from \(L^\infty(\Omega)\) strong into \(L^\infty(\Omega)\) weak*). If in addition \(K\) is strongly continuous from \(L^1(\Omega)\) into \(L^\infty(\Omega)\), then \((I+KF)^{-1}\) is strongly continuous from \(L^\infty(\Omega)\) into \(L^\infty(\Omega)\).

**Proof.** We shall prove a slightly stronger result. Let \(g_n\) be a bounded sequence in \(L^\infty(\Omega)\) such that \(g_n \to g\) a.e. on \(\Omega\). Let \(u_n = (I+KF)^{-1}g_n\) and let \(u = (I+KF)^{-1}g\). We are going to show that \(F(u_n) \to F(u)\) in \(L^1(\Omega)\).

We know, from the proof of Theorem 1, that \(\{u_n\}\) is bounded in \(L^\infty(\Omega)\) and there is some \(h \in L^1(\Omega)\) such that \(|F(u_n)| \leq h\) a.e. on \(\Omega\). Since
\[ \int_{\Omega} (u_n - u)(F(u_n) - F(u)) \, dx \leq \int_{\Omega} (g_n - g)(F(u_n) - F(u)) \, dx \]
and the right hand side goes to zero by Lebesgue's theorem, we can extract a subsequence such that
\[ (u_{n_k} - u)(F(u_{n_k}) - F(u)) \to 0 \quad \text{a.e. on } \Omega. \]
Consequently, \(F(u_{n_k}) \to F(u)\) a.e. on \(\Omega\) and hence \(F(u_{n_k}) \to F(u)\) in \(L^1(\Omega)\). By the uniqueness of the limit we conclude that \(F(u_n) \to F(u)\) in \(L^1(\Omega)\).

Using similar arguments, we can prove some variants of Theorem 1.

**Theorem 3.** Assume \(K\) is monotone, hemicontinuous and bounded from \(L^p'(\Omega)\) into \(L^p(\Omega)\). Assume \(f(x, r) : \Omega \times \mathbb{R} \to \mathbb{R}\) is continuous and nonincreasing in \(r\) for a.e. \(x \in \Omega\) and is measurable in \(x\) for all \(x \in \mathbb{R}\), and satisfies
\[ |f(x, r)| \leq c(x) + c_0 |r|^{p-1} \quad \text{a.e. } x \in \Omega, \text{ for all } r \in \mathbb{R} \]
where \(c \in L^p'(\Omega)\).

Then (1) has a unique solution \(u \in L^p(\Omega)\) for every \(g \in L^p(\Omega)\).

**Theorem 4.** Assume \(K\) is monotone, hemicontinuous from \(L^1(\Omega)\) into \(L^\infty(\Omega)\) and maps bounded sets of \(L^1(\Omega)\) into compact sets of \(L^\infty(\Omega)\).
Assume $f(x, r)$ is continuous in $r$ for a.e. $x \in \Omega$ and there exists $M$ such that
\[(f(x, r) - f(x, 0))r \geq 0 \text{ for a.e. } x \in \Omega \text{ and for all } |r| \geq M.\]
Suppose $f(x, r)$ is measurable in $x$ for all $r \in \mathbb{R}$ and for every constant $C$,
\[\int_{|r| \leq C} |f(x, r)| \text{ is integrable.}\]

Then (1) has a solution $u \in L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$.

The case of systems. Assume
(13) $K$ is monotone hemicontinuous and bounded from $L^1(\Omega; \mathbb{R}^n)$ into $L^\infty(\Omega; \mathbb{R}^n)$.
(14) $f(x, r): \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous in $r$ for a.e. $x \in \Omega$ and trimonotone in $r$, i.e., for a.e. $x \in \Omega$ and for any sequence $r_0, r_1, r_2, r_3 = r_0$ we have
\[\sum_{i=1}^3 (f(x, r_i), r_i - r_{i-1}) \geq 0\]
(for example, the gradient of a convex function is trimonotone, see [4]).
(15) $f(x, r)$ is measurable in $x$ for all $r \in \mathbb{R}$ and for every constant $C$
\[\int_{|r| \leq C} |f(x, r)| \text{ is integrable.}\]

**Theorem 5.** Under the assumptions (13), (14), (15), equation (1) has a unique solution $u \in L^\infty(\Omega; \mathbb{R}^n)$ for every $g \in L^\infty(\Omega; \mathbb{R}^n)$.

In order to bound $Fu$ in $L^1$, we use the following

**Lemma 2.** Assume (14) and (15) hold. Then for any constant $\rho > 0$, there exists $h_\rho \in L^1(\Omega)$ such that
\[\rho |f(x, r)| \leq (f(x, r) - f(x, 0), r) + h_\rho(x) \text{ for a.e. } x \in \Omega, \text{ all } r \in \mathbb{R}^n.\]

Uniqueness follows from the following

**Lemma 3.** Assume $B$ is continuous and trimonotone from a Hilbert space $H$ into itself. Let $u, v \in H$ be such that
\[(Bu - Bv, u - v) = 0.\]

Then $Bu = Bv$.

Along the same lines one can prove the following lemma which leads to stability results.

**Lemma 4.** Assume $B$ is trimonotone and Hölder continuous with exponent $\alpha \leq 1$ (i.e., $|Bu - Bv| \leq L|u - v|^{\alpha}$ for all $u, v \in H$).
Then there exists a constant $k>0$ such that
\[(Bu - Bv, u - v) \geq k |Bu - Bv|^{1+1/\alpha} \text{ for all } u, v \in H.\]

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