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SOME NEW RESULTS ABOUT HAMMERSTEIN EQUATIONS[1]

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Let Ω be a σ-finite measure space. Let K be a (nonlinear) montone operator and let $(Fu)(x)=f(x, u(x))$ be a Niemytski operator. We consider the Hammerstein type equation

$$u + KFu = g. \tag{1}$$

A detailed discussion and a complete bibliography about equation (1) can be found in [3]. The new feature about the results we present here is the fact that we do not assume any coercivity for F. When F is monotone and K maps $L^1(\Omega)$ into $L^\infty(\Omega)$, there is no growth restriction on F either (cf. Theorem 1). The monotonicity of F can be weakened when K is compact (cf. Theorem 4). Also some of these results are valid for systems in the case where F is the gradient of a convex function (cf. Theorem 5).

Assume

(2) K is a monotone hemicontinuous mapping from $L^1(\Omega)$ into $L^\infty(\Omega)$ which maps bounded sets into bounded sets,

(3) $f(x, r):\Omega\times \boldsymbol{R}\to\boldsymbol{R}$ is continuous and nondecreasing in r for a.e. $x\in\Omega$, and is integrable in x for all $r\in\boldsymbol{R}$.

THEOREM 1. *Under the assumptions* (2) *and* (3), *equation* (1) *has one and only one solution* $u\in L^\infty(\Omega)$ *for every* $g\in L^\infty(\Omega)$.

Uniqueness. Let u_1 and u_2 be two solutions of (1). By the monotonicity of K we get

$$\int_\Omega (u_1(x) - u_2(x))\cdot(f(x, u_1(x)) - f(x_1, u_2(x)))\,dx \leqq 0$$

which implies that $f(x, u_1(x))=f(x, u_2(x))$ a.e. on Ω and therefore by (1), $u_1=u_2$.

In proving existence of u we shall use the following

LEMMA 1. *Let X be a Banach space and let $K:X\to X^*$ and $F:X^*\to X$ be two monotone hemicontinuous operators. Let $\{u_n\}\subset X^*$, $\{v_n\}\subset X$ and*

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$\{w_n\} \subset X^*$ *be three sequences such that*

(4) u_n *converges to* u *in* X^* *for the weak* topology,*

(5) $F(u_n)$ *converges to* v *in* X *for the weak topology,*

(6) v_n *converges to* v *in* X *for the weak topology,*

(7) Kv_n *converges to* $g-u$ *in* X^* *for the weak* topology,*

(8) $\langle w_n, F(u_n)\rangle - \langle Kv_n, v_n\rangle \to 0$,

(9) $\langle g_n, F(u_n)\rangle \to \langle g, v\rangle$ *where* $g_n = u_n + w_n$.

Then $u + KFu = g$.

PROOF OF LEMMA 1. We have

$$\langle u_n - u, F(u_n)\rangle = \langle g_n - w_n - u, F(u_n)\rangle.$$

By the monotonicity of K we get

$$\langle Kv_n, v_n\rangle \geqq \langle Kv_n, v\rangle + \langle Kv, v_n - v\rangle$$

and thus

$$\liminf \langle Kv_n, v_n\rangle \geqq \langle g - u, v\rangle.$$

By (8) we have

$$\liminf \langle w_n, Fu_n\rangle \geqq \langle g - u, v\rangle.$$

Consequently, $\limsup \langle u_n - u, F(u_n)\rangle \leqq 0$. Since F is pseudomonotone (cf. [**1**]), we conclude that $v = Fu$ and $\langle u_n, F(u_n)\rangle \to \langle u, v\rangle$. Also $\langle Kv_n, v_n\rangle \to \langle g-u, v\rangle$ since $\langle w_n, F(u_n)\rangle = \langle g_n - u_n, F(u_n)\rangle \to \langle g, v\rangle - \langle u, v\rangle$. Thus

$$\lim \langle Kv_n, v_n - v\rangle = 0,$$

and again, since K is pseudomonotone, we conclude that $g - u = Kv = KFu$.

PROOF OF THEOREM 1. By a shift we can always assume that $f(x, 0) = 0$ and that $K0 = 0$ (note that (1) can be written as $u + \tilde{K}\tilde{F}u = \tilde{g}$, where $\tilde{F}v = Fv - F0$, $\tilde{K}v = K(v + F(0)) - KF0$ and $\tilde{g} = g - KF0$). Let Ω_n be an increasing sequence of finite measure subsets of Ω such that $\bigcup_n \Omega_n = \Omega$. Let χ_n be the characteristic function of Ω_n. Let F_n be F truncated by n, i.e.,

$$\begin{aligned} f_n(x, r) &= f(x, r) && \text{whenever } |f(x, r)| < n, \\ &= nf(x, r)/|f(x, r)| && \text{whenever } |f(x, r)| \geqq n. \end{aligned}$$

The equation

$$u_n + \chi_n K \chi_n F_n(u_n) = \chi_n g \tag{10}$$

has a solution.

Indeed the mapping $K_n : v \mapsto \chi_n K \chi_n v$ is monotone hemicontinuous from $L^2(\Omega)$ into itself.

On the other hand, the (multivalued) operator A defined on $L^2(\Omega)$ by

$$Av = \{w \in L^2(\Omega);\ v(x) = \chi_n(x) f_n(x, w(x)) \text{ a.e. on } \Omega\}$$

is maximal monotone in $L^2(\Omega)$ and $D(A)$ is bounded in $L^2(\Omega)$ ($|v|_{L^2} \leqq n(\text{meas}\, \Omega_n)^{1/2}$, $v \in D(A)$). Consequently, $R(A+K_n)=L^2(\Omega)$ (cf. [2]) and (10) has a solution.

Multiplying (10) through by $F_n(u_n)$ and using the monotonicity of K we get

$$\int_\Omega u_n \cdot F_n(u_n)\, dx \leqq \int_\Omega \chi_n g F_n(u_n)\, dx. \tag{11}$$

Let $C=2\|g\|_{L^\infty}$; we have

$$\begin{aligned}\int_\Omega u_n F_n(u_n)\, dx &= \int_{|un|\geqq C} u_n F_n(u_n)\, dx + \int_{|un|<C} u_n F_n(u_n)\, dx \\ &\geqq C\int_{|un|\geqq C} |F_n(u_n)|\, dx - C\int_{|un|<C} |F_n(u_n)|\, dx \\ &\geqq C\int_\Omega |F_n(u_n)|\, dx - 2C\int_{|un|<C} |F_n(u_n)|\, dx.\end{aligned}$$

Using (11) we obtain

$$\int_\Omega |F_n(u_n)|\, dx \leqq 4\int_{|un|\leqq C} |F_n(u_n)|\, dx \leqq 4\int_{|un|\leqq C} |f(x, u_n(x))|\, dx \leqq C'$$

by assumption (3).

Going back to (10), we conclude that $\{u_n\}$ remains bounded in $L^\infty(\Omega)$. Therefore, by assumption (3), there is some function $h \in L^1(\Omega)$ such that

$$|F_n(u_n)(x)| \leqq |f(x, u_n(x))| \leqq h(x) \quad \text{a.e. on } \Omega. \tag{12}$$

We apply now Lemma 1 with $v_n=\chi_n F_n(u_n)$, $w_n=\chi_n K v_n$, $g_n=\chi_n g$. By extracting a subsequence, we can always assume that

u_n converges to u weak* in $L^\infty(\Omega)$,

$F(u_n)$ converges to v weakly in $L^1(1)$,

v_n converges to v weakly in $L^1(\Omega)$,

g_n converges to g weak* in $L^\infty(\Omega)$.

Hence

w_n converges to $g-u$ weak* in $L^\infty(\Omega)$,

Kv_n converges to $g-u$ weak* in $L^\infty(\Omega)$.

It remains to verify (8) and (9). We have

$$\begin{aligned}\langle w_n, F(u_n)\rangle &= \int_\Omega \chi_n K v_n \cdot F(u_n)\, dx = \int_\Omega K v_n \chi_n F(u_n)\, dx \\ &= \int_\Omega K v_n \cdot v_n\, dx + \int_\Omega \chi_n K v_n (F(u_n) - F_n(u_n))\, dx.\end{aligned}$$

The last term can be bounded by

$$C\int_{|F(u_n)|>n} |Fu_n|\,dx \leqq C\int_{|h|>n} |h(x)|\,dx$$

which tends to zero as $n\to+\infty$ and (8) follows.

Finally (9) holds since

$$\langle g_n, F(u_n)\rangle = \int_\Omega \chi_n g F(u_n)\,dx = \int_\Omega gF(u_n)\,dx + \int_\Omega (\chi_n - 1)gF(u_n)\,dx,$$

and the last term goes to zero by Lebesgue's theorem.

THEOREM 2 (CONTINUOUS DEPENDENCE). *Under the assumptions* (2) *and* (3), *$F(I+KF)^{-1}$ is strongly continuous from $L^\infty(\Omega)$ into $L^1(\Omega)$ and $(I+KF)^{-1}$ is demicontinuous (from $L^\infty(\Omega)$ strong into $L^\infty(\Omega)$ weak*). If in addition K is strongly continuous from $L^1(\Omega)$ into $L^\infty(\Omega)$, then $(I+KF)^{-1}$ is strongly continuous from $L^\infty(\Omega)$ into $L^\infty(\Omega)$.*

PROOF. We shall prove a slightly stronger result. Let g_n be a bounded sequence in $L^\infty(\Omega)$ such that $g_n\to g$ a.e. on Ω. Let $u_n=(I+KF)^{-1}g_n$ and let $u=(I+KF)^{-1}g$. We are going to show that $F(u_n)\to F(u)$ in $L^1(\Omega)$.

We know, from the proof of Theorem 1, that $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and there is some $h\in L^1(\Omega)$ such that $|F(u_n)|\leqq h$ a.e. on Ω. Since

$$\int_\Omega (u_n - u)(F(u_n) - F(u))\,dx \leqq \int_\Omega (g_n - g)(F(u_n) - F(u))\,dx$$

and the right hand side goes to zero by Lebesgue's theorem, we can extract a subsequence such that

$$(u_{n_k} - u)(F(u_{n_k}) - F(u)) \to 0 \quad \text{a.e. on } \Omega.$$

Consequently, $F(u_{n_k})\to F(u)$ a.e. on Ω and hence $F(u_{n_k})\to F(u)$ in $L^1(\Omega)$. By the uniqueness of the limit we conclude that $F(u_n)\to F(u)$ in $L^1(\Omega)$.

Using similar arguments, we can prove some variants of Theorem 1.

THEOREM 3. *Assume K is monotone, hemicontinuous and bounded from $L^{p'}(\Omega)$ into $L^p(\Omega)$. Assume $f(x,r):\Omega\times\mathbf{R}\to\mathbf{R}$ is continuous and nonincreasing in r for a.e. $x\in\Omega$ and is measurable in x for all $x\in\mathbf{R}$, and satisfies*

$$|f(x,r)| \leqq c(x) + c_0\,|r|^{p-1} \quad \textit{a.e. } x\in\Omega, \textit{for all } r\in\mathbf{R}$$

where $c\in L^{p'}(\Omega)$.

Then (1) *has a unique solution $u\in L^p(\Omega)$ for every $g\in L^p(\Omega)$.*

THEOREM 4. *Assume K is monotone, hemicontinuous from $L^1(\Omega)$ into $L^\infty(\Omega)$ and maps bounded sets of $L^1(\Omega)$ into compact sets of $L^\infty(\Omega)$.*

Assume $f(x, r)$ is continuous in r for a.e. $x \in \Omega$ and there exists M such that

$$(f(x, r) - f(x, 0))r \geqq 0 \quad \textit{for a.e. } x \in \Omega \textit{ and for all } |r| \geqq M.$$

Suppose $f(x, r)$ is measurable in x for all $r \in \boldsymbol{R}$ and for every constant C,

$$\int_{|r| \leqq C} |f(x, r)| \quad \textit{is integrable.}$$

Then (1) *has a solution $u \in L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$.*

The case of systems. Assume

(13) K is monotone hemicontinuous and bounded from $L^1(\Omega; \boldsymbol{R}^n)$ into $L^\infty(\Omega; \boldsymbol{R}^n)$.

(14) $f(x, r): \Omega \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous in r for a.e. $x \in \Omega$ and trimonotone in r, i.e., for a.e. $x \in \Omega$ and for any sequence $r_0, r_1, r_2, r_3 = r_0$ we have

$$\sum_{i=1}^{3} (f(x, r_i), r_i - r_{i-1}) \geqq 0$$

(for example, the gradient of a convex function is trimonotone, see **[4]**).

(15) $f(x, r)$ is measurable in x for all $r \in \boldsymbol{R}$ and for every constant C

$$\int_{|r| \leqq C} |f(x, r)| \quad \text{is integrable.}$$

THEOREM 5. *Under the assumptions* (13), (14), (15), *equation* (1) *has a unique solution $u \in L^\infty(\Omega; \boldsymbol{R}^n)$ for every $g \in L^\infty(\Omega; \boldsymbol{R}^n)$.*

In order to bound Fu in L^1, we use the following

LEMMA 2. *Assume* (14) *and* (15) *hold. Then for any constant $\rho > 0$, there exists $h_\rho \in L^1(\Omega)$ such that*

$$\rho\,|f(x, r)| \leqq (f(x, r) - f(x, 0), r) + h_\rho(x) \quad \textit{for a.e. } x \in \Omega, \textit{ all } r \in \boldsymbol{R}^n.$$

Uniqueness follows from the following

LEMMA 3. *Assume B is continuous and trimonotone from a Hilbert space H into itself. Let $u, v \in H$ be such that*

$$(Bu - Bv, u - v) = 0.$$

Then $Bu = Bv$.

Along the same lines one can prove the following lemma which leads to stability results.

LEMMA 4. *Assume B is trimonotone and Hölder continuous with exponent $\alpha \leqq 1$ (i.e., $|Bu - Bv| \leqq L|u - v|^\alpha$ for all $u, v \in H$).*

Then there exists a constant $k>0$ *such that*

$$(Bu - Bv, u - v) \geqq k\,|Bu - Bv|^{1+1/\alpha} \quad \textit{for all } u, v \in H.$$

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