A complete convex hypersurface of a (separable) Hilbert space $H$ is a codimension one $C^\infty$ submanifold of $H$, which is complete as a metric subspace of $H$ and such that $M=\partial K$, where $K$ is a (closed) convex set with nonvoid interior. For each $p \in M$ let $v(p)$ be the unit normal vector which points to the interior of $K$. In this way we define the Gauss map $v: M \to \Sigma$ from $M$ into the unit sphere $\Sigma$ of $H$. This is a $C^\infty$ map and its derivative at each point $p \in M$ is selfadjoint. We say that $M$ bounds a half-line if there exists a half-line $\{p + tv; t \geq 0\}$ contained in the interior of $K$. In the finite dimensional case the condition that $M$ bounds a half-line is equivalent to that $M$ is unbounded. In the infinite dimensional case this is not true, as the following simple example shows. Let $A$ be a compact positive definite selfadjoint operator in $H$ and set $M=\{x \in H; \langle A(x), x \rangle = 1\}$. It is not difficult to prove that $M$ is an unbounded positively-curved convex hypersurface and that $M$ does not bound any half-line.

In this note we announce some properties of a complete convex hypersurface $M$ of a Hilbert space. Theorem A characterizes the three possible boundedness situations (bounded, unbounded and bounding a half-line, unbounded and bounding no half-line) in terms of the Gauss map of $M$. Theorem B gives a necessary and sufficient condition for $M$ to be a pseudo-graph (see definition below) over one of its tangent hyperplanes. Theorem C is the analogue of the Bonnet-Myers theorem for hypersurface of a Hilbert space. These results are part of my doctoral dissertation. I wish to thank my advisor Professor Manfredo do Carmo for suggesting these problems and for helpful conversations.

**THEOREM A.** Let $M$ be a complete convex hypersurface of a Hilbert space $H$. Then:

1. $M$ is bounded iff the Gauss map $v: M \to \Sigma$ is onto.
2. $M$ is unbounded and bounds a half-line iff the image of the Gauss map is contained in a hemisphere.
3. $M$ is unbounded and does not bound any half-line iff the image of the Gauss map is dense and has void interior.


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Before stating Theorem B, we need to rephrase slightly the definition of pseudograph given in [2]. A convex hypersurface $M$ is a pseudograph over the tangent hyperplane $F$ when:

(a) $M$ lies in one of the closed half-spaces determined by $F$,
(b) let $\pi: M \to F$ be the orthogonal projection and set $A = \pi(M)$. Then over the interior $\text{int} \ A$, $M$ is the graph of a $C^\infty$ function,
(c) for every $a \in A - \text{int} \ A$, $M \cap a^{-1}$ is a closed half-line,
(d) for every hyperplane $L$ above $F$, $M \cap L$ is bounded.

In the case that $M$ is finite dimensional, the above reduces to the definition given in [2].

**THEOREM B.** Let $M$ be a complete convex hypersurface of a Hilbert space $H$. Then $M$ is unbounded and $\text{int} (v(M)) \neq \emptyset$ iff $M$ is a pseudograph over one of its tangent hyperplanes $TM_p \neq M$.

**THEOREM C (THE BONNET-MYERS THEOREM FOR HILBERT HYPER­SURFACE).** Let $M$ be a complete connected hypersurface of a Hilbert space $H$. If the sectional curvatures of $M$ are all bounded away from zero (i.e. there exists a $\delta > 0$ such that for every $p \in M$ and every two-plane section $\sigma \subset TM_p$ one has $K(\sigma) \geq \delta$) then $M$ is bounded, the diameter $\rho$ of $M$ satisfies $\rho \leq \pi / \sqrt{\delta}$ and the Gauss map is a diffeomorphism.

**REMARK 1.** Theorem B should be compared with a theorem of H. H. Wu [2]. It should be remarked that Wu also proved that if $M$ is a complete convex hypersurface of $\mathbb{R}^n$, then $\text{int} (v(M)) = \text{int} (\bar{v}(M))$.

Theorem A shows that in the infinite dimensional case, this equality does not hold and we may have the extremal case in which $\text{int} (v(M)) = \emptyset$ and $\text{int} (\bar{v}(M)) = \Sigma$. This explains why we need the condition $\text{int} (v(M)) \neq \emptyset$ in Theorem B, in contrast with Wu's theorem, where no such condition is required.

**REMARK 2.** The hypothesis of Theorem B is implied by the following condition on the sectional curvature of $M$ (see [1]): The sectional curvatures of $M$ are everywhere nonnegative and at some point $p \in M$ are all bounded away from zero. Thus in the finite dimensional case, Theorem B reduces to Wu's theorem.

**REFERENCES**


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