IMMERSIONS OF THE CIRCLE AND EXTENSIONS TO ORIENTATION-PRESERVING MAPPINGS

BY ANTONIO O. FARIAS

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The author extends the results of [1] to the study of immersions of \( S^1 \) into a 2-dimensional, oriented manifold \( M \), and gives a characterization of those immersions which can be extended to a mapping of the closed disk with nonnegative Jacobian, in terms of geometric operations of growth. These are the analogue of the \( T \)-operators of [1]; they were introduced and studied by Titus in [2], and consist of displacement along the integral curves of a vector field toward the outside of the curve.

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1. Preliminaries. The reference for notation and terminology is the author’s paper [1]. The following additions and modifications will be used.

The phrases positive extension and positively extendable substitute proper extension and properly extendable, respectively. The fixed orientation of \( M \) is the 2-form \( \omega \). If \( \Lambda \) is a vector field on \( M \), then \( \Psi(\Lambda, z, t) \) denotes the integral curve of \( \Lambda \) through \( z \) at time \( t=0 \). All vector fields considered are divergence-free and have the property that \( \Psi(\Lambda, z, t) \) is defined for every \((z, t) \in M \times R^1 \).

Let \( \alpha: S^1 \rightarrow R^1 \) be nonnegative; define \( T=(\alpha, \Lambda) \) on the set \( C^\infty(S^1, M) \) by

\[
(T\alpha)(\theta) = \Psi(\Lambda, f(\theta), \alpha(\theta) \omega_{f(\theta)}[\Lambda_{f(\theta)}, f'(\theta)]).
\]

Denote by \( \mathcal{S} \) the semigroup generated by all such functions under composition; an element of \( \mathcal{S} \) will be called a \( T \)-operator.

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The effect of a $T$-operator on a curve $f:S^1 \to M$ is that of a finite number of growth operations, growth understood in the sense of motion along the integral curves of a vector field toward the outside of the oriented curve $f$.

A mapping $f:S^1 \to M$ is degenerate if it can be written as $f = g \circ h$, where $g:R^1 \to M$ is a diffeomorphism onto a submanifold of $M$. A $T$-mapping is one which can be written as $Tf_0$ with $T$ in $\mathcal{S}$ and $f_0$ degenerate.

2. Results.

**Theorem 1.** A normal mapping is extendable if and only if it is a $T$-mapping.

The if part was essentially proved by Titus [2, Theorem 1]. The only if part follows from Theorem 2 below.

**Theorem 2.** Every positively extendable mapping is a $T$-mapping.

In order to prove Theorem 2 we need to extend the results of [1] to immersions of $S^1$ into $M$. This is accomplished by first proving them for the case $M = S^2$ and then using universal covering arguments in the general case. In particular, the following results are proved:

**Theorem 3.** Every normal mapping has a neighborhood $\mathcal{U}$ in the $C^1$ topology of $C^\infty(S^1, M)$ with the following properties:

(a) If $g, h \in \mathcal{U}$ there are orientation-preserving diffeomorphisms $\varphi$ of $S^1$ and $\psi$ of $M$ such that $g \circ \varphi = \psi \circ h$.

(b) If there is a $g$ in $\mathcal{U}$ which is extendable, then every $h$ in $\mathcal{U}$ is also extendable.

**Proposition 1.** (1) Every normal, extendable mapping is positively extendable.

(2) The set of positively extendable mappings is open in the $C^1$ topology of $C^\infty(S^1, M)$.

(3) Let $F$ be a positive extension of $f$. There is a Riemann surface structure on $M$, a properly holomorphic $W:D^+ \to M$, an orientation-preserving homeomorphism $H:D^- \to D^-$ and an open set $U$ containing $S^1$ such that $W \circ H = F$ on $U$ and the restriction of $H$ to $U$ is a diffeomorphism into $R^2$.

In the proof of part (3) use is made of the following result in general topology.

**Proposition 2.** Let $X$ be a Hausdorff space and $A$ a closed subspace with empty interior. Let $F:X \to Y$ be a continuous mapping such that the restriction of $F$ to $X - A$ is a homeomorphism into $Y$. Then, $F(A) \cap F(X - A) = \emptyset$. If, further, $X$ is locally compact, $Y$ is Hausdorff and $F$ is a
local homeomorphism from $A$ to $F(A)$, then $F$ is a homeomorphism of $X$ into $Y$.

Details and proofs will appear elsewhere.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SÃO PAULO, SÃO PAULO, BRAZIL