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IMMERSIONS OF THE CIRCLE AND EXTENSIONS TO ORIENTATION-PRESERVING MAPPINGS[1]

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The author extends the results of [1] to the study of immersions of S^1 into a 2-dimensional, oriented manifold M, and gives a characterization of those immersions which can be extended to a mapping of the closed disk with nonnegative Jacobian, in terms of geometric operations of growth. These are the analogue of the T-operators of [1]; they were introduced and studied by Titus in [2], and consist of displacement along the integral curves of a vector field toward the outside of the curve.

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1. **Preliminaries.** The reference for notation and terminology is the author's paper [1]. The following additions and modifications will be used.

The phrases *positive extension* and *positively extendable* substitute *proper extension* and *properly extendable*, respectively. The fixed orientation of M is the 2-form ω. If Λ is a vector field on M, then $\Psi\{\Lambda, z, t\}$ denotes the integral curve of Λ through z at time $t=0$. All vector fields considered are divergence-free and have the property that $\Psi\{\Lambda, z, t\}$ is defined for every $(z, t) \in M \times R^1$.

Let $\alpha: S^1 \to R^1$ be nonnegative; define $T=(\alpha, \Lambda)$ on the set $C^\infty(S^1, M)$ by

$$(1.1) \qquad (Tf)(\theta) = \Psi\{\Lambda, f(\theta), \alpha(\theta)\omega_{f(\theta)}[\Lambda_{f(\theta)}, f'(\theta)]\}.$$

Denote by $\mathscr{S}$ the semigroup generated by all such functions under composition; an element of $\mathscr{S}$ will be called a T-operator.

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[1] Part of this research was done while the author was a Visiting Professor at Universidade Federal de Pernambuco (Brazil).

The effect of a T-operator on a curve $f: S^1 \to M$ is that of a finite number of growth operations, growth understood in the sense of motion along the integral curves of a vector field toward the outside of the oriented curve f.

A mapping $f: S^1 \to M$ is *degenerate* if it can be written as $f = g \circ h$, where $g: R^1 \to M$ is a diffeomorphism onto a submanifold of M. A *T-mapping* is one which can be written as Tf_0 with T in $\mathcal{S}$ and f_0 degenerate.

2. **Results.**

THEOREM 1. *A normal mapping is extendable if and only if it is a T-mapping.*

The *if* part was essentially proved by Titus [**2**, Theorem 1]. The *only if* part follows from Theorem 2 below.

THEOREM 2. *Every positively extendable mapping is a T-mapping.*

In order to prove Theorem 2 we need to extend the results of [**1**] to immersions of S^1 into M. This is accomplished by first proving them for the case $M = S^2$ and then using universal covering arguments in the general case. In particular, the following results are proved:

THEOREM 3. *Every normal mapping has a neighborhood $\mathcal{U}$ in the C^1 topology of $C^\infty(S^1, M)$ with the following properties*:

(a) *If $g, h \in \mathcal{U}$ there are orientation-preserving diffeomorphisms φ of S^1 and ψ of M such that $g \circ \varphi = \psi \circ h$.*

(b) *If there is a g in $\mathcal{U}$ which is extendable, then every h in $\mathcal{U}$ is also extendable.*

PROPOSITION 1. (1) *Every normal, extendable mapping is positively extendable.*

(2) *The set of positively extendable mappings is open in the C^1 topology of $C^\infty(S^1, M)$.*

(3) *Let F be a positive extension of f. There is a Riemann surface structure on M, a properly holomorphic $W: D^- \to M$, an orientation-preserving homeomorphism $H: D^- \to D^-$ and an open set U containing S^1 such that $W \circ H = F$ on U and the restriction of H to U is a diffeomorphism into R^2.*

In the proof of part (3) use is made of the following result in general topology.

PROPOSITION 2. *Let X be a Hausdorff space and A a closed subspace with empty interior. Let $F: X \to Y$ be a continuous mapping such that the restriction of F to $X - A$ is a homeomorphism into Y. Then, $F(A) \cap F(X - A) = \varnothing$. If, further, X is locally compact, Y is Hausdorff and F is a* *local homeomorphism from A to $F(A)$, then F is a homeomorphism of X into Y.*

Details and proofs will appear elsewhere.

References

1. Antonio O. Farias, *Orientation-preserving mappings, a semigroup of geometric transformations and a class of integral operators*, Trans. Amer. Math. Soc. **167** (1972), 279–289. MR **45** #4440.

2. Charles J. Titus, *Transformation semigroups and extensions to sense-preserving mappings*, 1970/71 Preprint Series no. 35, Matematisk Institut, Aarhus Universitet, Denmark.

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