STABLE MINIMAL SURFACES
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Let $M \subset R^3$ be a minimal surface. A domain $D \subset M$ is an open connected set with compact closure $\bar{D}$ and such that its boundary $\partial D$ is a finite union of piecewise smooth curves. We say that $D$ is stable if $D$ is a minimum for the area function of the induced metric, for all variations of $\bar{D}$ which keep $\partial D$ fixed. In this note we announce the following estimate of the "size" of a stable minimal surface. We will denote by $S^2$ the unit sphere of $R^3$.

**Theorem.** Let $g: M \subset R^3 \rightarrow S^2$ be the Gauss map of a minimal surface $M$ and let $D \subset M$ be a domain. If area $g(D) < 2\pi$ then $D$ is stable.

**Remark.** The estimate is sharp, as can be shown, for instance, by considering pieces of the catenoid bounded by circles $C_1$ and $C_2$ parallel to and in opposite sides of the waist circle $C_0$. By choosing $C_1$ close to $C_0$ and $C_2$ far from $C_0$, we may obtain examples of unstable domains whose area of the spherical image is bigger than $2\pi$ and as close to $2\pi$ as we wish.

**Remark.** The theorem implies that if the total curvature of $D$ is smaller than $2\pi$, then $D$ is stable. The theorem is however stronger since we only use the area of the spherical image and the total curvature is equal to this area counting multiplicity.

**Remark.** Our theorem is related to a result of A. H. Schwarz (see, for instance, [3, p. 39]).

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Below we present a brief sketch of the proof. A complete proof along with further results will appear elsewhere.

**Sketch of the proof.** Let $\Delta$ and $K$ be the laplacian and the gaussian curvature of $M$, respectively, in the induced metric. Assume that $D$ is not stable. It follows from the Morse index theorem [4] that there

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1 Partially supported by N.S.F. and C.N.Pq.
exists a domain $D' \subset D$ and a $C^\infty$-function $u$ on $D'$ such that $u > 0$ on
$D'$, $u \equiv 0$ on $\partial D'$, and $\Delta u - 2uK = \Delta u = 0$. Since $g$ is antiholomorphic and $D'$
is compact, there are only finitely many points $p_1^0, \ldots, p_m^0$ where $K = 0$. Thus, in $D' = \bigcup \{p_j^0\}, j = 1, \ldots, m$, $g$ is a local diffeomorphism, and for
each $q \in g(D')$, $g^{-1}(q) \cap \bar{D}' = \{p_1, \ldots, p_n\}$ is finite. We then define a
function $f$ on $g(D')$ by

$$f(q) = \sum_i u(p_i), \quad i = 1, \ldots, n.$$  

It is easily seen that $f$ is not identically zero and $f \equiv 0$ on $\partial (g(D'))$. Furthermore, it can be proved that $f$ is continuous in $g(D')$ and differentiable
in $g(D') - g(\partial D' \cup \{p_1^0, \ldots, p_m^0\})$.

A crucial point in the proof is to show that

$$(1) \quad \int_{g(D')} |\nabla f|^2 \, dS \leq 2 \int_{g(D')} f^2 \, dS,$$

where $dS$ is the element of area of $S^2$. This is accomplished by decom­posing $g(D')$ in a suitable way and, using the fact that $g$ is antiholomor­phic, by lifting parts of the above integrals into $D'$, where they can be more easily computed. The remaining parts are then estimated and this
yields the required inequality.

From now on, it will be convenient to denote by $\lambda_1^T$ the first eigenvalue
for the spherical laplacian of a domain $T \subset S^2$. It follows from (1) that

$$(2) \quad \lambda_1^{(D')} \leq 2.$$  

A further important point in the proof is to show that, given a domain
$T \subset S^2$, it is possible to deform it, keeping its area fixed and not in­creasing its first eigenvalue, in such a way that it will eventually lie inside
an open "cap" (i.e., a domain in $S^2$ bounded by an equator). To do this, we adapt, for functions and domains on a sphere, the process of circular
symmetrization about an axis known in the plane (see [2, p. 193]). Using the ideas of [2] it is not difficult to show that such a process keeps the
area of $T$ fixed and does not increase $\lambda_1^T$. The main point is now to prove
that by successive applications of this process, the boundary of $T$ conver­ges to a parallel of the sphere. This requires estimating some geometri­cal quantities which appear in the process of symmetrization.

It follows that $g(D')$ can be symmetrized into a domain $g(D')^*$, such
that $g(D')^*$ is properly contained in a hemisphere $H \subset S^2$ and $\lambda_1^{(D')} \leq \lambda_1^{(D')^*} \leq 2$. Since $\lambda_1^H = 2$, and a proper inclusion increases strictly the first
eigenvalue, we obtain that $\lambda_1^{(D')^*} > 2$, and this contradiction proves the
theorem.

REMARK. The above process of symmetrization can be used to give
a proof of the isoperimetric inequality on the sphere. Our convergence
argument also contains a proof of a spherical version of the Faber-Krahn inequality (see [1, p. 413]).

REMARK. The arguments which lead to inequality (1) can also be used to prove that if $g(D) \subset T \subset S^2$ and $\lambda_1^T > 2$ then $D$ is stable.

REFERENCES


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