ON VITALI-HAHN-SAKS TYPE THEOREMS

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In recent years extensive work has been done on the Vitali-Hahn-Saks theorem and its relatives. Seever [13] considered the question of extending the Vitali-Hahn-Saks theorem to the case where the domain is a Boolean algebra which is not necessarily sigma complete. Brooks and Jewett [2] established results for a strongly bounded map defined on a Boolean sigma algebra of sets with values in a Banach space. Further generalizations to group-valued set functions have been studied by the Poznán school (see [5], [6], [7], [8], [9], [11], [12]). The work of all these authors is generalized herein to the case of strongly bounded maps defined on Boolean algebras with the Seever property and taking values in a Banach space. Some applications other than those considered herein and the final generalization to the group-valued case can be found in [10].

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1. Notation and definitions. A Boolean algebra \( S \) has the property (I) if and only if for any sequences \( \{x_n\} \) and \( \{y_n\} \) in \( S \) satisfying \( x_n \leq y_m \) for all \( n, m \), there exists \( x \in S \) such that \( x_n \leq x \leq y_n \) for all \( n \). This condition is equivalent to the condition: given any sequences \( \{a_n\} \) and \( \{b_n\} \) in \( S \) satisfying \( a_n \wedge a_m = 0, b_n \wedge b_m = 0 \) for \( n \neq m \) and \( a_n \wedge b_m = 0 \) for all \( n, m \), there exists an element \( a \in S \) such that \( a \geq a_n \) and \( b_n \wedge a = 0 \) for all \( n \).

Unless signified otherwise, \( S \) will be used in this paper to denote a Boolean algebra with the property (I). The symbol \( X \) denotes a Banach space and \( X^* \) its Banach space dual.

A finitely additive \( \mu: S \to X \) is bounded whenever there exists \( M > 0 \) such that \( \|\mu(b)\| \leq M \) for all \( b \in S \); \( \mu \) is said to be strongly bounded if \( \|\mu(e_n)\| \to 0 \) as \( n \to \infty \) for each disjoint sequence \( e_1, \ldots, e_n, \ldots \) of elements in \( S \). A sequence \( \mu_n: S \to X, n = 1, 2, \ldots \), is uniformly strongly bounded if for each disjoint sequence \( \{e_n\} \subseteq S \), \( \lim_n \sup_k \|\mu_k(e_n)\| = 0 \). By grouping it is easy to see that if \( \mu \) is strongly bounded and \( \{e_n\} \subseteq S \) is disjoint, then \( \sum_{n=1}^{\infty} \mu(e_n) \) is an unconditionally convergent series in \( X \). A map \( \mu: S \to X \) is


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countably additive if for every disjoint sequence \( \{e_n\} \subset B \) with \( \bigvee_n e_n \in B \), the equality \( \mu(\bigvee_n e_n) = \sum_n \mu(e_n) \) holds. The semivariation of \( \mu \) on \( b \in B \), denoted by \( \|\mu\|(b) \), is defined to be \( \sup\{\|\mu(a)\| : a \in B, a \leq b\} \). It is easily shown that \( \mu : B \to X \) is strongly bounded if and only if \( \|\mu\| : B \to [0, \infty) \) is strongly bounded (though \( \|\mu\| \) need not be additive).

2. Main results.

**Theorem 1.** Let \( \mu_n : B \to X \) be finitely additive and strongly bounded for \( n = 1, 2, \cdots \). If \( \lim_n \mu_n(e) = 0 \) for each \( e \in B \), then \( \{\mu_n : n \in N\} \) is uniformly strongly bounded.

**Proof.** Suppose not. Then there exists a sequence \( \{e_n\} \) of disjoint elements of \( B \), a number \( \varepsilon > 0 \), and a sequence \( m_1 < m_2 < m_3 < \cdots \) of positive integers (to simplify notation, assume \( m_n = n \)) such that for each \( n \in N \), \( \|\mu_n(e_n)\| > 4\varepsilon \).

Let \( i_1 = 1 \). Partition the set \( N \setminus \{1\} \) into an infinite number of infinite disjoint sets \( \pi_n^1, n = 1, 2, 3, \cdots \). Utilizing property (I) we can choose a sequence \( f_n^1, n = 1, 2, \cdots \), of disjoint elements in \( B \) such that:

- \( f_n^1 \geq e_i \) for all \( i \in \pi_n^1, n = 1, 2, \cdots \);
- \( f_n^1 \wedge e_{i_1} = 0, n = 1, 2, \cdots \);
- \( f_n^1 \wedge e_j = 0 \) for all \( j \in (N \setminus \{1\}) \setminus (\bigcup_{i=1}^n \pi_i^1) \).

As \( \|\mu_{i_1}\|(f_n^1) \to 0 \) \((n \to \infty)\) there exists an \( n_1 \in N \) such that \( \|\mu_{i_1}\|(f_n^1) < \varepsilon \).

Choose \( i_2 > i_1 \) and \( \|\mu_{i_1}(e_{i_1})\| < \varepsilon /4 \). Partition the set \( \pi_{n_1}^1 \setminus \{i_1\} \) into an infinite number of infinite disjoint sets \( \pi_n^2, n = 1, 2, \cdots \). Again by property (I) there exists a sequence \( f_n^2, n = 1, 2, \cdots \), of disjoint elements in \( B \) such that:

- \( f_n^2 \geq e_i \) for all \( i \in \pi_n^2, n = 1, 2, \cdots \);
- \( f_n^2 \wedge (e_{i_1} \vee e_{i_2}) = 0, n = 1, 2, \cdots \);
- \( f_n^2 \wedge e_j = 0 \) for all \( j \in (\pi_{n_1}^1 \setminus \{i_1\}) \setminus (\bigcup_{i=1}^n \pi_i^2) \).

There exists an integer \( n_2 \in N \) such that \( \|\mu_{i_2}\|(f_n^2) < \varepsilon \).

Choose \( i_3 > i_2 \) and \( \|\mu_{i_2}(e_{i_2})\|, \|\mu_{i_2}(e_{i_1})\| < \varepsilon /8 \). Proceed in this fashion to obtain a sequence \( f_n^k = f_k, k = 1, 2, \cdots \), of elements of \( B \) and a sequence \( i_1 < i_2 < \cdots \) of positive integers such that:

1. \( f_n \geq e_{i_k}, k > n \);
2. \( f_n \wedge e_{i_k} = 0, 1 \leq k \leq n \);
3. \( \|\mu_{i_1}(f_n)\| < \varepsilon, n = 1, 2, \cdots \);
4. \( \|\mu_{i_k}(e_{i_1})\| < \varepsilon /2^n, 1 \leq k < n \);
5. \( \|\mu_{i_k}(e_{i_1})\| > 4\varepsilon, n = 1, 2, \cdots \).

Let \( h_n = f_n \vee (\bigvee_{k=1}^n e_{i_k}) \). Then \( h_n \geq e_{i_k} \) for all \( n, k \). Choose \( c \in B \) such that

\[
\tag{6} h_n \geq c \geq e_{i_n} \quad \text{for all } n.
\]
Noticing that $\mu_n(c) = \mu_n(h_n - e_n) - \mu_n(h_n - c) + \mu_n(e_n)$, we have

$$\|\mu_n(c)\| \leq \|\mu_n(e_n)\| - \|\mu_n(h_n - e_n)\| - \|\mu_n(h_n - c)\|$$

$$= \|\mu_n(e_n)\| - \|\mu_n\left[\left(\bigvee_{k=1}^{n} e_{ik}\right) \wedge e_i\right]\|$$

$$- \|\mu_n\left[\left(\bigvee_{k=1}^{n} e_{ik}\right) \wedge c'\right]\|,$$

which by (2) is

$$\geq \|\mu_n(e_n)\| - \|\mu_n(f_n \wedge e'_n)\|$$

$$- \|\mu_n\left[\left(\bigvee_{k=1}^{n} e_{ik}\wedge e'_{in}\right)\right]\| - \|\mu_n\left[\left(\bigvee_{k=1}^{n} e_{ik} \wedge c'\right)\right]\|.$$

Applying (2), (6) and the disjointness of the $e_{ik}$'s yields

$$\geq \|\mu_n(e_n)\| - \|\mu_n(f_n)\|$$

$$- \|\mu_n(e_n)\| - \cdots - \|\mu_n(e_{in-1})\| - \|\mu_n(f_n \wedge c')\|,$$

which by (5), (3) and (4) is $>4\varepsilon - \varepsilon - (n-1)e/2^n - \varepsilon \geq \varepsilon$. Since $\|\mu_n(c)\| > \varepsilon$ holds for infinitely many $n$, $\lim_n \mu_n(c) \rightarrow 0$, a contradiction.

The proofs of some of the corollaries yielded by Theorem 1 are, for the most part, minor alterations to proofs presented elsewhere; in these cases the appropriate references are given.

**Corollary 1 [2, Corollary 1.2].** Let $\mu_n: \mathcal{B} \rightarrow X$ be finitely additive and strongly bounded for $n = 1, 2, \ldots$. If $\lim_n \mu_n(e) = \mu(e)$ exists for each $e \in \mathcal{B}$, then $\mu$ is strongly bounded and the $\mu_n$, $n = 1, 2, \ldots$, are uniformly strongly bounded.

**Corollary 2.** Let $\mu_n: \mathcal{B} \rightarrow X$ be countably additive for $n = 1, 2, \ldots$. If $\lim_n \mu_n(e) = \mu(e)$ exists for each $e \in \mathcal{B}$, then $\mu$ is countably additive and the $\mu_n$, $n = 1, 2, \ldots$, are uniformly countably additive.

**Corollary 3 [3, Theorem 1.6].** Let $X$ be any separable Banach space and let $\mu: \mathcal{B} \rightarrow X$ be bounded and finitely additive. Then $\mu$ is strongly bounded.

Another corollary is the following result proved differently by N. J. Kalton in an unpublished manuscript.

**Corollary 4.** Let $X$ be a weakly compactly generated Banach space and let $\mu: \mathcal{B} \rightarrow X$ be bounded and finitely additive. Then $\mu$ is strongly bounded.
PROOF. Let \( \{e_n\} \) be a disjoint sequence in \( \mathcal{B} \) and let \( [\mu(e_n)] = X_0 \) denote the closed linear span of \( \{\mu(e_n): n \in \mathbb{N}\} \). Then \( X_0 \) is a separable subspace of the weakly compactly generated space \( X \); hence by a result of Amir and Lindenstrauss [1, Lemma 4], there is a separable subspace \( Y \) of \( X \) such that \( X_0 \subseteq Y \) and \( Y \) is complemented in \( X \). Suppose \( P: X \to Y \) is the projection. Then Corollary 1.2 yields \( P \circ \mu(e_n) \to 0 \), \( n \to \infty \). But \( P \circ \mu(e_n) = \mu(e_n) \) for each \( n \). Therefore, \( \mu \) is strongly bounded.

COROLLARY 5 [4, COROLLARY 5]. Let \( \mu_n: \mathcal{B} \to X \) be strongly bounded for \( n=1, 2, \cdots \). Suppose \( \mu(e) = \text{weak-limit}_n \mu_n(e) \) exists for each \( e \in \mathcal{B} \). Then \( \mu \) is strongly bounded.

PROOF. The boundedness of \( \mu \) follows from the Banach-Steinhaus theorem and Corollary 1.1 applied to the functions \( f \mu_n, f \mu \) where \( f \in X^* \). For each \( n \) let \( B_n = \mu_n(\mathcal{B}) \) and let \( B = \bigcup_n B_n \). By the definition of \( \mu \) and Mazur's theorem we have \( \mu(\mathcal{B}) \subseteq Y \). We claim that \( Y \) is weakly compactly generated.

For each \( n \), let \( M_n = \sup \| \mu_n(b) \| : b \in \mathcal{B} \). Let \( B = \bigcup_n B_n/(n \cdot M_n) \). The closed linear span of \( B \) is \( Y \) and \( B \) is relatively weakly compact. To see the last assertion, let \( \{y_n\} \) be a sequence in \( B \). Since each \( \mu_n \) is strongly bounded, \( B_n \), and hence \( B_n/(n \cdot M_n) \), is relatively weakly compact [14]. So if \( \{y_n\} \) returns infinitely often to one of the \( B_n/(n \cdot M_n) \)'s, we can extract a weakly convergent subsequence. If \( \{y_n\} \) does not return infinitely often to any \( B_n/(n \cdot M_n) \) then there exist strictly increasing sequences \( (m_k) \) and \( (n_k) \) of positive integers such that \( y_{m_k} \in B_{n_k}/(n_k \cdot M_{n_k}) \) for each \( k \). It follows that \( \| y_{m_k} \| \leq 1/n_k \to 0 \) as \( k \to \infty \). Thus \( \{y_n\} \) has a norm convergent subsequence.

With the proof proceeding as in [2, Theorem 3] we have the following Vitali-Hahn-Saks theorem.

THEOREM 2. Let \( \mu_n: \mathcal{B} \to X \) be finitely additive and strongly bounded, for \( n=1, 2, \cdots \). Suppose \( \nu \) is a nonnegative monotone set function defined on \( \mathcal{B} \) and each \( \mu_n \ll \nu \). Assume that \( \lim_n \mu_n(e) \) exists for each \( e \in \mathcal{B} \). Then \( \lim_{r(e) \to 0} \| \mu_n(e) \| = 0 \) uniformly in \( n \).

REFERENCES


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