A LOWER ESTIMATE FOR EXPONENTIAL SUMS

BY C. A. BERENSTEIN\(^1\) AND M. A. DOSTAL

Communicated by François Treves, October 8, 1973

1. Introduction. In this note we present two theorems on exponential sums (see Theorems 1 and 2 below). Although seemingly unrelated, both results are motivated by the study of a certain type of lower estimates of exponential sums in the complex domain. Thus while Theorem 2 is related to the validity of this estimate for all \textit{discrete} exponential sums\(^2\), Theorem 1 essentially says that even a milder estimate of this kind does not hold for a whole class of \textit{continuous} exponential sums (i.e. for certain Fourier transforms).

In addition to the usual notation of the theory of distributions (cf. [2], [3], [7]), the following symbols will be used throughout this note. Given a distribution \(\Phi \in \mathcal{D}'(\mathbb{R}^n)\), the symbol \([\Phi]\) \((\{\Phi\}\) resp.) denotes the convex hull of the support of \(\Phi\) (singular support of \(\Phi\), resp.). For \(A \subset \mathbb{R}^n\), \(h_A\) is the supporting function of \(A\), i.e. \(h_A(\lambda) = \sup_{x \in A} \langle x, \lambda \rangle\), \(\lambda \in \mathbb{R}^n\). For \(\zeta \in \mathbb{C}^n\) and \(r > 0\), \(\Delta(\zeta; r)\) is the closed polydisk with center \(\zeta\) and radius \(r\); and, if \(g(\zeta')\) is any continuous function on \(\Delta(\zeta; r)\), we shall write

\[
|g(\zeta)|_r = \max_{\zeta' \in \Delta} |g(\zeta')|.
\]

2. Indicators of smooth convex bodies.

\textbf{Definition.} Let \(\Phi \in \mathcal{D}'(\mathbb{R}^d)\) be such that

\[
\{\Phi * \Psi\} = \{\Phi\} + \{\Psi\} \quad (\forall \Psi \in \mathcal{D}')
\]

Then \(\Phi\) will be called a \textit{good convolutor}.

The relationship of being a good convolutor to the solvability of the convolution equation \(\Phi * u = f\) in the appropriate distribution spaces was discovered by L. Hörmander [7], and since then it was discussed by several authors (for references, cf. [2, Chapter I]). However, it is usually not easy to decide whether a given distribution \(\Phi\) is a good convolutor or not.

\textit{AMS (MOS) subject classifications} (1970). Primary 33A10, 32A15, 47G05.

\textit{Key words and phrases}. Exponential-polynomials, Fourier transforms, several complex variables, convolution equations.

\(^1\) The first author was supported in part by the U.S. Army Research Office (Durham)

\(^2\) And more generally, for all exponential polynomials.
Moreover, few good convolutors are known, and as Theorem 1 below will indicate, even distributions of a very simple nature may fail to be good convolutors.

It can be shown [4, Proposition 2] that the following condition on $\hat{\Phi}$ is sufficient for $\Phi$ to be a good convolutor:

**CONDITION** $(R_w)$. There exist constants $t \geq 0$, $r > 0$, $c > 0$ and $A$ real (all depending on $\Phi$) so that (cf. (1))

$$|\hat{\Phi}(\zeta)|_r \geq c(1 + |\xi|)^r \exp(h_t(\phi)(\eta))$$

for all $\zeta = \xi + i\eta \in \mathbb{C}^n$ such that $|\xi| \geq t$ and $|\eta| > t \log(1 + |\xi|)$.

Since any distribution $\Phi$ with finite support satisfies condition $(R_w)$ (cf. [4, Proposition 6]), we thus obtain a result of Hörmander [7], [8], according to which all distributions with finite support are good convolutors. This in turn can be used to prove the following statement (cf. [4, Proposition 6]):

Let $P$ be an arbitrary compact convex polyhedron in $\mathbb{R}^n$ and $\chi_P$ the distribution defined by the characteristic function of $P$. Then $\chi_P$ satisfies condition $(R_w)$, hence $\chi_P$ is a good convolutor. The same conclusion holds for the surface measure $\chi_{\partial P}$ of density 1, i.e.

$$\chi_{\partial P}(\phi) = \int_{\partial P} \phi(x) \, ds_x, \quad (\phi \in \mathcal{E})$$

where $ds_x$ is the surface element.

It seems natural to ask whether this proposition holds for smooth convex bodies $P$ as well. At the first glance it seems that it does. Indeed, if, for instance, $P$ is any ellipsoid in $\mathbb{R}^n$, then the distribution $\Phi = \chi_P$ satisfies the following weaker version of (2) (cf. the concluding remark in [5]):

$$(2^*) \{\Psi\} \subseteq \{\Phi * \Psi\} - \{\Phi\} \quad (\forall \Psi \in \mathcal{E}).$$

Therefore, it is rather surprising that this particular $\Phi$ is not a good convolutor [5, Proposition 4]. The following theorem sheds more light on this peculiar situation.

**THEOREM 1.** Let $P$ be a convex body in $\mathbb{R}^n$ ($n > 1$) with a $C^\infty$-boundary $\partial P$. Moreover, it is assumed that the Gaussian curvature of $\partial P$ never vanishes, i.e. $K(x) > 0$ for every $x \in \partial P$. Then neither $\chi_P$ nor $\chi_{\partial P}$ is a good convolutor.

**REMARK.** Both assumptions on $\partial P$ (i.e. smoothness and $K > 0$) can be substantially relaxed.

The proof of Theorem 1 is based on a detailed study of the asymptotic behavior of the functions $\hat{\chi}_P$ and $\hat{\chi}_{\partial P}$ in the complex domain. For $\xi$ real, estimates of this kind were previously derived by numerous authors (cf. [9], [10], [11] and the references given in [10], [11]). However, for our
purposes these estimates must be sharpened. As an illustration, consider
the case of the convex surface $S = \partial P$. Given $\zeta = \xi + i\eta \in \mathbb{C}^n$ with $\xi \neq 0$, write $r = |\xi|$ and consider $\zeta = r\omega + i\eta$ with $\omega$ fixed. Let $x^j = (x^j_1, \ldots , x^j_n) \in S$ ($j = 0, 1$) be the points

$$x^j_v = \partial h_S(( -1)^j \zeta)/\partial x^j_v \quad (v = 1, \ldots , n).$$

Fix arbitrarily the open subsets $S_k$ ($k = 0, 1, 2$) of $S$ so that $S = \bigcup_k S_k$, $S^0 \cap S^1 = \varnothing$, $x^j \in S^0 \setminus S^1$ ($j = 0, 1$). Then for any $q > n/2$ and $r > 0$ there exist positive numbers $a_j, b_j$ and $c_r$ such that

$$\hat{\chi}_S(\eta) = (1 - i)^{n-1} \left( \frac{\pi}{2} \right)^{n-1} r^{(1-n)/2} \sum K(x^j)^{-1/2} \exp(-i\langle x^j, \zeta \rangle)
\begin{align*}
&+ I_1 + I_2 + I_3; \\
|I_1(\eta)| &\leq r^{-n/2} (1 + |\eta|)^s \sum a_j \exp(\langle x^j, \eta \rangle), \\
|I_2(\eta)| &\leq r^{-q} (1 + |\eta|)^{2q} \sum b_j \exp[h_S(\eta)], \\
|I_3(\eta)| &\leq c_r r^{-n} (1 + |\eta|) \exp[h_S^d(\eta)],
\end{align*}
(4)

where $\sum = \sum_{j=0,1}$. Formula (4) combined with a result of H"ormander [8] yields Theorem 1 for $\chi^P$. Asymptotic expansions similar to (4) hold for $\chi^P$ as well as for the Fourier transforms of certain measures with non-constant density.

3. The discrete case. Generalization of Ritt's theorem. In this part we shall consider finite exponential sums, and more generally, exponential polynomials in several complex variables. If $H$ is an exponential polynomial, i.e. a function of the form

$$H(\zeta) = \sum_{j=1}^{s} h_j(\zeta) \exp(\langle \theta_j, \zeta \rangle) \quad (\zeta \in \mathbb{C}^n)$$

(5)

with complex frequencies $\theta_j \in \mathbb{C}^n$ and polynomial coefficients $h_j$, the greatest common divisor of the $h_j$'s, $d_H=(h_1, \ldots , h_s)$, will be called the content of $H$. Moreover, we shall write $\mathcal{C}_H(\zeta) = \max_j \text{Re}(\theta_j, \zeta)$. Henceforth an exponential sum will mean a function of the form (5) with all coefficients $h_j$ constant. The following lower estimate of exponential polynomials was proved in [3], [5]:

(R$_a$) Given an exponential polynomial $H$ and an arbitrary $\varepsilon > 0$, there exists $C = C(\varepsilon, H) > 0$ such that for every $\zeta \in \mathbb{C}^n$ and any $f$ analytic in $\Delta(\zeta; \varepsilon)$,

$$|f(\zeta)| \exp(\mathcal{C}_H(\zeta)) \leq C |f(\zeta)H(\zeta)|. \quad (6)$$

* Obviously, estimate (R$_a$) is much stronger than (R$_{\omega}$).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
In this section we shall discuss the following

**Question.** Let \( F \) and \( G \) be exponential polynomials in \( n \) variables such that the function \( H = F/G \) is entire. What can be said about the structure of \( H \)? In particular, when is \( H \) an exponential polynomial?

Simple examples show that \( H \) need not be an exponential polynomial (e.g., \( n = 1, F = \sin \zeta, G = \zeta \)). On the other hand, if \( F \) and \( G \) are exponential sums in one variable such that \( H \) is entire, then, according to a theorem of Ritt [12], \( H \) is also an exponential sum. Different proofs of Ritt's theorem were given by H. Selberg, P. D. Lax and A. Shields (cf. the references in [12], [13]). In particular, Shields [13] proves that \( H \) is an exponential polynomial as long as it is entire and \( G \) is an exponential sum. He also mentions that, according to an unpublished result of W. D. Bowsma, the last assumption may be replaced by \( d_G = 1 \). Finally, Avanissian and Martineau [1] generalized the original Ritt's theorem to arbitrary \( n \geq 1 \). The following theorem contains all these results as special cases. Moreover, it shows that the above counterexample is in a certain sense the best possible:

**Theorem 2.** Let \( F, G, H \) be as above (\( n \geq 1 \) arbitrary). Then there exists an exponential polynomial \( E \) and a polynomial \( Q \) such that \( H = E/Q \). Hence we may assume \( (d_E, Q) = 1 \). Then \( E \) and \( Q \) are determined uniquely* and \( Q \) divides \( d_G \).

The starting point for the proof of Theorem 2 is the following assertion: Let \( f, g, h \) be the analytic functionals whose Fourier-Borel transforms are \( F, G, H \) respectively. Then \( h \) is carried by the polyhedron defined by \( C_F - C_G \). This in turn follows from \( (R_0) \).

The proofs together with applications of the above theorems will appear elsewhere.

**BIBLIOGRAPHY**

1. V. Avanissian, Oral communication, 1970.

* Up to a constant multiple.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742

DEPARTMENT OF MATHEMATICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY 07030

INSTITUTO DE MATEMATICA PURA E APlicada, RIO DE JANEIRO, BRAZIL