A function \( f(z) \) analytic in the unit disk is said to belong to the Bergman space \( A^p \) \((0 < p < \infty)\) if 
\[
\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r \, dr \, d\theta < \infty.
\]
It is clear that \( A^p \) contains the Hardy space \( H^p \) of analytic functions for which 
\[
\lim_{\rho \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.
\]
We adopt the convention that \( A^\infty = H^\infty \), the space of bounded analytic functions in the disc.

Assuming that \( f(0) \neq 0 \), we list the zeros of \( f \) in order of nondecreasing modulus: 
\[
0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots < 1.
\]
We repeat \( z_i \) according to the multiplicity of the zero of \( f \) at \( z_i \). The sequence \( \{z_i\} \) is called the zero set of \( f \). If \( f \in A^p \) (resp. \( H^p \)), then \( z_i \) will be called an \( A^p \) (resp. \( H^p \)) zero set. It has long been known that \( H^p \) zero sets \((0 < p < \infty)\) are completely characterized by the condition 
\[
\sum_{k=1}^{\infty} 1/|z_k| < \infty.
\]
(Equivalently, \( \sum_{k=1}^{\infty} 1 - |z_k| < \infty \).) In particular, the condition is independent of \( p \).

Our results show that the situation for \( A^p \) zero sets is considerably more complex.

**Lemma 1.** If \( \{z_k\} \) is an \( A^p \) zero set \((0 < p < \infty)\), then
\[
\frac{1}{\prod_{k=1}^{N} |z_k|} = O(N^{1/p}).
\]

**Corollary.** If \( \{z_k\} \) is an \( A^p \) zero set \((0 < p < \infty)\), then for each \( \varepsilon > 0 \),
\[
\sum_{k=1}^{\infty} (1 - |z_k|) \left( \log \frac{1}{1 - |z_k|} \right)^{-1-\varepsilon} < \infty.
\]

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), let \( S_n^{(p)} = \sum_{n=1}^{N} |a_k|^p \), \( p > 0 \).

**Lemma 2.** If \( S_n^{(p)} = O(N^\alpha) \) for some \( \alpha \geq 1 \), then \( f \in A^p \) for all \( p < 2/\alpha \).

**Lemma 3.** For some \( p \), \( 1 \leq p \leq 2 \), suppose that \( \sum_{n=1}^{\infty} N^{-p} S_n^{(p)} < \infty \) and 
\[
N^{1-p} S_n^{(p)} = O(1).
\]
Then \( f \in A^p \), \( 1/p + 1/p' = 1 \).

Lemma 1 is proved by an application of Jensen’s theorem. Lemmas 2 and 3 follow from corresponding coefficient conditions, after a summation by parts. In particular, Lemma 3 is a consequence of the fact that

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Theorem 1. Let $0 < p < q \leq \infty$. Then there exists an $A^p$ zero set which is not an $A^q$ zero set.

Sketch of Proof. Let $f(z) = \prod_{k=0}^{\infty} 1 + uz^k$, where $b$ is an integer greater than 2, and $u$ is a positive constant. Using the notation of the lemmas, one verifies that:

1. Every partial product for $f(z)$ is a partial sum of its Taylor series.
2. If $N = \sum_{k=0}^{b^s-1} b^s$, $S_N^{(p)} = (1 + u^p)^s$.
3. If $u > 1$, if $N = \sum_{k=0}^{b^s-1} b^s$, and if $\{z_i\}$ are the ordered zeros of $f$, then $\prod_{i=1}^{\infty} 1/|z_i| = u^s$.

From these facts, and from Lemmas 1, 2 and 3, we conclude that:

4. If $b \leq 1 + u^s$, then $f \in A^p$ for all $p < 2 \log b / \log(1 + u^s)$. (Also, in this case, $f \notin A^q$.)
5. If $1 + u^s \leq b^{s-1}$ for some $s$, $1 < s \leq 2$, $f \in A^p$ for all $p < s'$, where $1/s + 1/s' = 1$.
6. If $u > 1$, the zero set of $f$ is not the zero set of any function in $A^q$ for $q > \log b / \log u$.

An examination of (4), (5) and (6) shows that if $0 < p < q \leq \infty$, $u$ and $b$ may always be chosen to yield a function $f$ in $A^p$ whose zero set is not an $A^q$ zero set.

Theorem 2. For $0 < p < \infty$, the union of two $A^p$ zero sets is not in general an $A^p$ zero set.

To prove Theorem 2, we choose one of the functions $f \in A^p$ constructed in Theorem 1, with the parameter $u > 1$. We choose a positive integer $N$ and require that each zero of $f$ be repeated $N$ times. For $N$ sufficiently large we obtain a sequence which, by Lemma 1, cannot be an $A^q$ zero set.

We state two corollaries to the above theorems, both of which again contrast sharply with $H^p$ theory.

Corollary (to Theorem 1). It is not possible to represent an arbitrary $A^1$ function as the product of two functions in $A^2$, one of them nonvanishing.

Corollary (to Theorem 2). Consider the operator $M_z$ of multiplication by $z$ on $A^2$ (a weighted unilateral shift). There exist two nontrivial closed invariant subspaces of $M_z$ whose intersection is trivial.