BILINEAR FORMS AND CYCLIC GROUP ACTIONS

BY J. P. ALEXANDER, G. C. HAMRICK AND J. W. VICK

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In a recent paper [2] Conner and Raymond have given an approach to
the study of smooth cyclic group actions which employs rational bilinear
forms. If $K^{2n-1} = \partial B^{4n}$ bounds a compact oriented smooth manifold,
there is a symmetric nonsingular rational bilinear form on the image of
$H^{2n}(B, K; Q) \rightarrow H^{2n}(B; Q)$ which represents an element $w(B)$ in $W(Q)$, the
rational Witt ring. Denoting the signature of this form by $\text{sgn}(B)$ and the
unit of $W(Q)$ by $1$, the \textit{peripheral invariant} of $K$,

$$\text{per}(K) = w(B) - \text{sgn}(B) \cdot 1,$$

lies in the kernel of the signature homomorphism $\Phi: W(Q) \rightarrow \mathbb{Z}$ and is
independent of the choice of $B$. In [2] there is associated with any orienta-
tion preserving diffeomorphism $(T, M)$ of prime period $p$ on a closed
manifold an element of the kernel of $\Phi$ which we denote by $q(T, M)$, an
invariant of the equivariant bordism class which vanishes on fixed point
free actions. Using the peripheral invariant, Conner and Raymond
computed $q(T, M)$, for $p=2$ or $3$, in terms of the fixed point information.
The fundamental problem posed in [2] is the extension of this result to all
primes.

In this paper we give the general formula for all primes and apply it to
establish relationships between the index of $M$ and the index of the fixed
set. The essence of the proof is a group isomorphism between the kernel of
$\Phi$ and $\bigoplus_p W(Z_p)$ where $W(Z_p)$ is the Witt group of the field $Z_p$ and the
sum ranges over all primes. Using this isomorphism, we establish a
relation between the peripheral invariant and the linking form which
enables us to extend the definition of $\text{per}(K)$ to any closed oriented
$(4k-1)$-manifold.

1. \textbf{Bilinear forms.} Let $B$ Fin denote the semigroup of isomorphism
classes of symmetric nonsingular bilinear forms on finite abelian groups
taking values in $Q/Z$. Denote by $W^s(Z)$ the semigroup of stable equiv-
ance classes of nondegenerate integral bilinear forms on finitely

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generated free abelian groups, where stability means that we are allowed
to add the form $x^2$ or the form $-x^2$ without altering the equivalence class.
Let $W^s(Q)$ be the corresponding group of rational bilinear forms. Kneser
and Puppe [5] have shown that there is a one-to-one correspondence
between $W^s(Z)$ and $B \text{ Fin}$ (see also [3], [6], [7], [8]).

The composition $B \text{ Fin} \to W^s(Z) \to W^s(Q)$ is clearly onto and generates
an equivalence relation $\sim$ on $B \text{ Fin}$, which we refer to as rational equivalence of finite forms. Suppose $(\lambda, G)$ is a finite form in $B \text{ Fin}$ and $K \subseteq H \subseteq G$ are subgroups such that

$$H = \{x \in G \mid \lambda(x, y) = 0 \text{ for all } y \in K\} = K^\perp,$$

the annihilator of $K$. $\lambda$ induces a nonsingular form $\lambda'$ on $H/K$.

1.1. Theorem. If $G$, $H$, $K$, $\lambda$ and $\lambda'$ are as given above, $(\lambda, G) \sim (\lambda', H/K)$ in $B \text{ Fin}$. Conversely, if $(\lambda, G)$ is rationally trivial, there is a sub­
group $H \subseteq G$ such that $H = H^\perp$.

We denote by $\mathcal{W}$ the Grothendieck group generated by $B \text{ Fin}$ modulo
the subgroup generated by all forms $(\lambda, G)$ such that there is a subgroup
$H \subseteq G$ with $|H|^2 = |G|$ and $\lambda(H, H) = 0$.

1.2. Theorem. If $W(Z_p)$ denotes the Witt group of nonsingular bilinear
forms over $Z_p$, the inclusion induces an isomorphism of groups, $\oplus_p W(Z_p) \to \sim \mathcal{W}$, where the sum ranges over all primes $p$.

We can summarize the above results in the following corollary [9].

1.3. Corollary. There is a sequence of group isomorphisms

$$W^s(Q) \approx \mathcal{W} \approx \bigoplus_p W(Z_p),$$

and since $W^s(Q)$ may be identified with the kernel of the signature homo­
morphism $\Phi: W(Q) \to (Z)$, there is an isomorphism of groups (but not of rings)

$$W(Q) \approx W(R) \oplus \left( \bigoplus_p W(Z_p) \right).$$

A form $\lambda: Z_p \times Z_p \to Q/Z$ with $\lambda(1, 1) = b/p$ is completely determined up
to isomorphism by $[b] \in Z_p^*/Z_p^{**}$, the multiplicative group of units
modulo squares. Denote the form $\lambda(1, 1) = b/p$ by $\langle b \rangle_p$. As abelian groups
we have for $p \equiv 3$ (mod 4), $W(Z_p) \approx Z_4$ generated by $\langle 1 \rangle_p$; for $p \equiv 1$ (mod 4),
$W(Z_p) \approx Z_2 \oplus Z_2$ generated by $\langle 1 \rangle_p$ and $\langle a \rangle_p$ where $a$ is not a square mod $p$, and
$W(Z_2) \approx Z_2$ generated by $\langle 1 \rangle_2$. The integral form corresponding to
$\langle 1 \rangle_p$ is the $1 \times 1$ matrix $(p)$. There is a concise algorithm for constructing
the matrix for the integral form corresponding to $\langle a \rangle_p$. 

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2. **Prime order actions.** Let $M^{4n-1}$ be a closed oriented smooth manifold and let $G$ be the torsion subgroup of $H^{2n}(M; \mathbb{Z})$. Recall the definition of the linking form $\lambda$ on $M$: If $x, y \in G$ then $x = \beta(z)$ for some $z \in H^{2n-1}(M; \mathbb{Q}/\mathbb{Z})$ where $\beta$ is the Bockstein homomorphism. Then $\lambda(x, y) = \langle z \cup y, [M] \rangle$ in $Q/\mathbb{Z}$ defines a nonsingular bilinear pairing on $G$. Denote the corresponding element of $\mathcal{W}$ by $\lambda(M)$.

Suppose that $M^{4n-1} = \partial B^{4n}$ where $B^{4n}$ is a compact oriented smooth manifold. For $i: M \to B$ the inclusion map, define

$$H = \{x \in G \mid x = i^*(\xi) \text{ for some } \xi \in H^{2n}(B; \mathbb{Z})\},$$

and

$$K = \{x \in G \mid x = i^*(\xi) \text{ for some } \xi \in \text{Tor } H^{2n}(B; \mathbb{Z})\}.$$

Then $K$ is isomorphic to $G/H$. Note that a necessary condition for $B^{4n}$ to be a rational disk is that $|G|$ be a square, since in this case $H = K$. This gives information on a question posed in [1].

2.1. **Corollary.** If $T$ is a smooth diffeomorphism of prime period on $S^{2k-1}$ with fixed set $M^{4n-1}$ such that the order of $\text{Tor}(H^{2n}(M; \mathbb{Z}))$ is not a square, then $T$ cannot be smoothly extended to $D^{2k}$.

2.2. **Lemma.** Under the linking form $(\lambda, G)$, $H = \{x \in G \mid \lambda(x, y) = 0 \text{ for all } y \in K\} = K^\perp$.

2.3. **Theorem.** If $M^{4n-1} = \partial B^{4n}$ as above, then under the isomorphism $W^s(\mathcal{Q}) \approx \mathcal{W}$,

$$\text{per}(M) = -\lambda(M).$$

By taking this equation as the definition, the peripheral invariant may be extended to all closed oriented $(4n-1)$-manifolds (in fact, using techniques analogous to those for the index, it can be extended to compact manifolds with boundary).

2.4. **Corollary.** $\text{per}(M)$ is defined for all closed oriented $(4n-1)$-manifolds and is an invariant of the oriented homotopy type of $M$.

2.5. **Corollary.** If $T$ is a smooth diffeomorphism of prime period on $S^{2k-1}$ with fixed set $M^{4n-1}$ having $\text{per}(M) \neq 0$, then $T$ cannot be smoothly extended to $D^{2k}$.

The lens spaces give an interesting set of examples for examining the peripheral invariant as well as for applications. The quotient of the action of $\mathbb{Z}_p$ on $S^{4n-1}$ given by $T(z_1, \cdots, z_{2n}) = (\alpha^s z_1, \cdots, \alpha^s z_{2n})$, where $\alpha = e^{2\pi i/p}$ and $(r_j, p) = 1$, gives the lens space $L^{4n-1}(p; r_1, \cdots, r_{2n})$. For each $j$ choose an integer $l_j$ with $l_j \cdot r_j = 1 \text{ mod } p$. Let $l = l_1 \cdot l_2 \cdot \cdots \cdot l_{2n}$.

2.6. **Proposition.** The linking form on this lens space is given by $\lambda(L^{4n-1}(p; r_1, \cdots, r_{2n})) = \langle l \rangle_p$. 

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Conner and Raymond [2] have defined an invariant for smooth periodic group actions that fits nicely into this setting. Let \((T, M^{2n})\) be an orientation-preserving diffeomorphism of odd prime period \(p\) on a closed manifold. There is a symmetric, nonsingular bilinear form on \(H^{2n}(M; \mathcal{Q})\) given by \(f(x, y) = p \cdot (x \cup y, [M]) \in \mathcal{Q}\). The restriction of \(f\) to the fixed vectors defines an element \(w(T, M) \in W(\mathcal{Q})\), whose signature we write as \(\text{sgn}(M/T)\). The invariant defined in [2] which we have denoted by \(q(T, M)\) is defined by
\[
q(T, M) = w(T, M) - \text{sgn}(M/T) \cdot 1.
\]
Since this lies in the kernel of \(\Phi: W(\mathcal{Q}) \to \mathbb{Z}\), we view it as an element of \(\mathcal{W}\). One of the principal results of [2] is the determination of this invariant for \(p = 2\) or \(3\).

2.7. THEOREM (CONNER AND RAYMOND [2]). If \(p = 3\), or if \(p = 2\) and \(T\) is weakly complex, then \(q(T, M) = \text{sgn}(F) \cdot (1)^k\) where \(F\) is the fixed set.

Let \(N\) be an equivariant tubular neighborhood of \(F\) in \(M\). The relationship between this invariant and the peripheral invariant may be stated [2] as
\[
q(T, M) = p \otimes w(N) - \text{sgn}(N) \cdot 1 - \text{per}(\partial N/T)
\]
where tensoring a rational form with \(p\) corresponds to multiplying each entry in its matrix by \(p\).

Now suppose that \(F_0^{2k}\) is a component of the fixed set and \(S^{2m-1} \to \partial N_0 \to F_0^{2k}\) is the equivariant sphere bundle over \(F_0\). The quotient under \(T\) is the lens space bundle \(L_0^{2m-1} \to \partial N_0/T \to F_0^{2k}\).

An argument involving spectral sequences shows:

2.8. THEOREM. The local fixed point information is given by
\[
p \otimes w(N_0) - \text{sgn}(N_0) \cdot 1 - \text{per}(\partial N_0/T) = \text{sgn}(F_0^{2k}) \cdot \lambda(L_0^{2m-1}).
\]

2.9. COROLLARY. (a) For \(p \equiv 3 \pmod{4}\) give the normal bundle to \(F\) a complex structure in which all eigenvalues are of the form \(\alpha^k\) where \(k\) is a square mod \(p\). Then if \(F\) is given the orientation consistent with the orientation of \(M\),
\[
q(T, M) = \text{sgn}(F) \cdot \langle 1 \rangle_p.
\]

(b) For \(p \equiv 1 \pmod{4}\) orient \(F\) arbitrarily. Let \(F_1\) be the union of those components of \(F\) in which the corresponding lens space has \(\lambda(L) = \langle 1 \rangle_p\) and \(F_2\) the union of the remaining components. Then
\[
q(T, M) = \text{sgn}(F_1) \cdot \langle 1 \rangle_p + \text{sgn}(F_2) \cdot \langle a \rangle_p.
\]
2.10. Corollary. Suppose \((T, M)\) is as above and \(T^*\) is the identity on \(H^{2n}(M; \mathcal{O})\). Then with \(F\) oriented as in (2.9),

(a) For \(p \equiv 3 (\text{mod} \ 4)\), \(\text{sgn}(M) \equiv \text{sgn}(F) \mod 4\).

(b) For \(p \equiv 1 (\text{mod} \ 4)\), \(\text{sgn}(M) \equiv \text{sgn}(F) \equiv \text{sgn}(F_x) \mod 2\).

(Note that a unimodular form of dimension less than \(p\) has no isometries of order \(p\).)

Related results comparing the index of \(M\) to the index of \(F\) have been obtained by Lowell Jones [4] using completely different methods. It may be seen by simple examples that the relation in (a) is the best possible. We have evidence that in (b) there may be a \(\mathbb{Z}_4\) invariant. In fact for \(p = 5\) an argument using the Atiyah-Singer index theorem shows that \(\text{sgn}(M) \equiv \text{sgn}(F) \mod 4\).

References