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RESTRICTED APPROXIMATION AND INTERPOLATION

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There has been much interest recently in questions of approximating or interpolating continuous functions by polynomials which are subject to certain constraints. Among the problems considered are those of (a) monotone approximation [**5**], [**6**], [**10**], [**11**], (b) comonotone approximation [**8**], [**9**], (c) piecewise monotone interpolation [**3**], [**12**], [**13**], and (d) copositive approximation. We outline the problems below and state our results. The proofs will appear elsewhere.

(a) *Monotone approximation.* Let P_n be the set of all algebraic polynomials of degree $\leqq n$. Let $K=\{0\leqq k_1<k_2<\cdots<k_q\}$ be a set of integers, let $\varepsilon_i=\pm 1$, $i=1, 2, \cdots, q$, and let $f\in C^j[0, 1]$, where $j\geqq k_q$. Suppose that $\varepsilon_i f^{(k_i)}(x)\geqq 0$ for all $x\in[0, 1]$, $i=1, 2, \cdots, q$. (More generally, we may assume only that $\varepsilon_i\Delta^{(k_i)}f\geqq 0$, $i=1, 2, \cdots, q$, for nondifferentiable functions.) Let

$$E_n(f; K)=\inf\{\|f-p\|:p\in P_n, \varepsilon_i p^{(k_i)}(x)\geqq 0, x\in[0, 1], i=1, 2, \cdots, q\}.$$

$E_n(f; K)$ is called the *degree of monotone approximation* of f. Lorentz and Zeller [**6**] have shown that if $K=\{1\}$, and $f\in C[0, 1]$ is nondecreasing, then $E_n(f; K)=O(\omega(f; 1/n))$.

THEOREM 1. *Let $f\in C[0, 1]$ and assume that $\varepsilon_i\Delta^{(k_i)}f(x)>0$ in $[0, 1]$, $i=1, 2, \cdots, q$. Then $E_n(f; K)=O(\omega(f; 1/n))$.*

THEOREM 2. *Let $f\in C[0, 1]$, let k be an integer, and assume that $\Delta^{(k)}f(x)\geqq 0$ in $[0, 1]$. Then for any $\varepsilon>0$ there exists $d(k, \varepsilon)$ such that $E_n(f; K)\leqq d(k, \varepsilon)\omega(f; 1/n^{1-\varepsilon})$, for n sufficiently large.*

(b) *Comonotone approximation.* Let $f\in C[0, 1]$ be a function having a finite number of local extrema. Such a function is said to be *piecewise monotone*. The local extrema are called the *peaks* of f. Two functions f and g are said to be *comonotone* on $[0, 1]$ if f and g are increasing and decreasing on exactly the same subintervals of $[0, 1]$. Let f be piecewise monotone on $[0, 1]$ and let

$$E_n^*(f)=\inf\{\|f-p\|:p\in P_n, p \text{ comonotone with } f\}.$$

$E_n^*(f)$ is called the *degree of comonotone approximation* of f.

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THEOREM 3 [9]. *Let* $f \in C^{j+l}[0, 1]$, *with peaks at* $x_1, x_2, \cdots, x_{l-1}$. *Then there exists* d_j *such that* $E_n^*(f) \leqq d_j \|f^{(j+l)}\|/n^j$ *for* $n > 2(j+l)$.

THEOREM 4 [9]. *Let* $f \in C^{j+l}[0, 1]$, *with peaks at* $x_1, x_2, \cdots, x_{l-1}$. *Then there exists* $r_{j,l}$, *depending on* j *and* $x_1, x_2, \cdots, x_{l-1}$, *such that* $E_n^*(f) \leqq r_{j,l}\|f^{(j+l)}\|/n^{j+l-2}$ *for* $n > 4(j+l+1)$.

THEOREM 5. *Let* $f \in C[0, 1]$, *with peaks at* $x_1, x_2, \cdots, x_{l-1}$. *Then for any* $\varepsilon > 0$ *there exists* $b(l, \varepsilon)$ *such that* $E_n^*(f) \leqq b(l, \varepsilon)\omega(f; 1/n^{1-\varepsilon})$ *for* n *sufficiently large.*

(c) *Piecewise monotone interpolation.* Let $X = \{0 = x_0 < x_1 < \cdots < x_k = 1\}$ and let $Y = \{y_0, y_1, \cdots y_k\}$ be a set of real numbers such that $y_{i-1} \neq y_i$, $i = 1, 2, \cdots, k$. It is a result of Wolibner [12], Kammerer [3], and Young [13] that there exists a polynomial p such that $p(x_i) = y_i$, $i = 0, 1, \cdots, k$, and p is monotone on each of the intervals (x_{i-1}, x_i), $i = 1, 2, \cdots, k$. Such a polynomial is said to interpolate *piecewise monotonely*. The smallest degree of a polynomial that interpolates the values Y at the points X piecewise monotonely is called the *degree of piecewise monotone interpolation of* Y *with respect to* X, and is denoted by $N(X; Y)$.

Let S_j, $j = 0, 1, 2, \cdots$, be the set of all piecewise monotone functions, $f \in \mathrm{Lip}_1 1$, with j peaks. The *degree of comonotone approximation to* S_j is defined by

$$E_n^*(S_j) = \sup\{E_n^*(f) : f \in S_j\}.$$

By Theorem 5, $\lim_{n\to\infty} E_n^*(S_j) = 0$. The smallest degree n such that $E_n^*(S_j) < \delta$ is denoted $n_j(\delta)$. With given data X and Y we associate a piecewise linear function $L(x) = L(X; Y; x)$ defined by $L(x_i) = y_i$, $i = 0, 1, \cdots, k$. Let j be the number of peaks of L.

Let

$$\Delta = \Delta(Y) = \min_i |y_i - y_{i-1}| \quad \text{and} \quad M = M(X; Y) = \max_i \frac{y_i - y_{i-1}}{x_i - x_{i-1}}.$$

THEOREM 6. $N(X; Y) \leqq n_j(\Delta/12M)$.

THEOREM 7. *If* $y_0 < y_1 < \cdots < y_k$, *then there exists an absolute constant* A *such that* $N(X; Y) \leqq AM/\Delta$.

Let $G(M; \Delta) = \{(X; Y) : M(X; Y) \leqq M, \Delta(Y) \geqq \Delta\}$ and let $N(G) = \sup\{N(X; Y) : (X; Y) \in G\}$.

THEOREM 8. *Let* $y_0 < y_1 < \cdots < y_k$. *Then there exist constants* $c_1, c_2 > 0$ *such that* $c_1 M/\Delta \leqq N(G) \leqq c_2 M/\Delta$.

Theorem 8 shows that Theorem 7 is essentially unimprovable in a class-wide sense.

(d) *Copositive approximation.* f and g are *copositive* on $[0, 1]$ if f and g are nonnegative and nonpositive on exactly the same subintervals of $[0, 1]$. Let $f \in C[0, 1]$ and let

$$\bar{E}_n(f) = \inf\{\|f - p\| : p \in P_n, p \text{ copositive with } f\}.$$

$\bar{E}_n(f)$ is called the *degree of copositive approximation* of f. A piecewise monotone function f with peaks at $x_1, x_2, \cdots, x_l$ is said to be *proper* if for any $\varepsilon>0$ there exists $\delta>0$ such that $|(f(x)-f(y))/(x-y)| \geqq \delta$ for $x \neq y$ and x and y in $[x_i+\varepsilon, x_{i+1}-\varepsilon]$, $i=1, 2, \cdots, l-1$. Using the results in [**8**] we derive the following.

THEOREM 9. *Let $f \in C[0,1]$ be a proper piecewise monotone function such that $f \in \mathrm{Lip}_M 1$. Let f have peaks at $0=x_1<x_2<\cdots<x_l=1$. Suppose that $f(x_i) \neq 0$, $i=1, 2, \cdots, l$. Then there exists d, depending on f such that $\bar{E}_n(f) \leqq d\omega(f; 1/n) \leqq dM/n$ for n sufficiently large.*

All of the preceding theorems involve approximation and interpolation by algebraic polynomials. The following results are for Muntz polynomials. Recent studies of the approximating properties of Muntz polynomials are to be found in [**2**], [**4**], [**7**].

In Theorems 10–13, we let $T=\{t_0, t_1, \cdots\}$ be a sequence of nonnegative real numbers with the properties: (i) $\lim_{i\to\infty} t_i=\infty$, (ii) $\sum_{t_i\neq 0} 1/t_i=\infty$.

THEOREM 10. *Let $f \in C[0, 1]$ have sign changes at $x_1<x_2<\cdots<x_{k-1}$. Suppose that $0, 1, \cdots, 2k-1 \in T$. Then for any $\varepsilon>0$ there exists $p(x)=\sum_{i=0}^n a_i x^{t_i}$ such that p is copositive with f on $[0, 1]$ and $\|f-p\|<\varepsilon$.*

THEOREM 11. *Let $f \in C[0, 1]$ be a piecewise monotone function with peaks at $0=x_0<x_1<\cdots<x_k=1$. Suppose that $0, 1, \cdots, 2k \in T$. Then for any $\varepsilon>0$ there exists $p(x)=\sum_{i=0}^n a_i x^{t_i}$ such that p is comonotone with f and $\|f-p\|<\varepsilon$.*

THEOREM 12. *Let $X=\{0=x_0<x_1<\cdots<x_m=1\}$ and let $Y=\{y_0, y_1, \cdots, y_m\}$ be a set of real numbers such that $y_{i-1}\neq y_i$, $i=1, 2, \cdots, m$. Let $L(X)$ be the piecewise linear function defined by $L(x_i)=y_i$, $i=0, 1, \cdots, m$, and suppose that L has $k+1$ peaks, $0=z_0<z_1<\cdots<z_k=1$. Suppose that $0, 1, \cdots, 2k \in T$. Then there exists $p(x)=\sum_{i=0}^n a_i x^{t_i}$ such that $p(x_i)=y_i$, $i=0, 1, \cdots, m$, and p is monotone in each of the subintervals (x_{i-1}, x_i), $i=1, 2, \cdots, m$.*

THEOREM 13. *Let $f \in C[0, 1]$, let $k \geqq 1$ be an integer, and suppose that $\Delta^{(k)} f(x) \geqq 0$ in $[0, 1]$. Suppose that $0, 1, \cdots, k \in T$. Then for any $\varepsilon>0$ there exists $p(x)=\sum_{i=0}^n a_i x^{t_i}$ such that $p^{(k)}(x) \geqq 0$ and $\|f-p\|<\varepsilon$.*

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