1. Introduction. Let $\mathcal{N}_n(q)$ (resp. $\mathcal{N}_n^{or}(q)$) be the bordism group of $n$-dimensional smooth manifolds with arbitrary (resp. oriented) $q$-plane fields, and let $\Omega_n(q)$ and $\Omega_n^{or}(q)$ denote the corresponding groups based on oriented manifolds. In this paper we present a method which allows us in many cases to determine these groups. We use the forgetful homomorphism $f_{\mathcal{N}}: \mathcal{N}_n(q) \to \mathcal{N}_n(BO(q))$ (resp. $f_{\mathcal{N}}: \mathcal{N}_n^{or}(q) \to \mathcal{N}_n(BSO(q))$, resp. $f_{\Omega}: \Omega_n^{or}(q) \to \Omega_n(B(S)O(q)))$, which assigns to the bordism class of a $q$-plane field the bordism class of (a classifying map of) the underlying vector bundle. Our point of departure is the following observation. If $\xi$ is a $q$-dimensional vector bundle over an $n$-manifold $M$ and $n \geq 2q - 3$, then it is always possible to find a vector bundle homomorphism $h: \xi \to TM$ which is injective outside of a $(q - 1)$-dimensional submanifold $S$ of $M$, and such that the kernel of $h$ is 1-dimensional at every point of $S$. We investigate the behavior of $h$ at such a singularity and obtain criteria as to when it is possible to cancel $S$ without getting out of the original bordism class.

If $M$ is closed and $\xi$ is isomorphic to a $q$-dimensional subbundle of $TM$, then the element $TM - \xi$ in the $K$-theory of $M$ can be represented by an $(n-q)$-dimensional bundle, and hence the class $[M, \xi]$ in the bordism of $B(S)O$ satisfies the following vanishing condition:

(V) all Whitney numbers of $[M, \xi]$ containing some $w_i(TM - \xi)$, $i > n - q$, as a factor, vanish.

Conversely we obtain

**Theorem 1.** Let $n > 2q - 2$. Then under all four orientedness assumptions $[M, \xi]$ lies in the image of $f_{\mathcal{N}}$ if and only if condition (V) is satisfied. Furthermore, the kernel as well as the cokernel of $f_{\mathcal{N}}$ are finite groups consisting entirely of elements of order 2.

A stable version of the first statement for the case of $\mathcal{N}_n(BO(q))$ has previously been obtained by R. Stong [11] by other methods.
COROLLARY 1. $\mathcal{B}n_n(q)$ and $\mathcal{B}n_n(q)$ are finite vector spaces over $\mathbb{Z}_2$. $\mathcal{B}O_n(q)$ and $\mathcal{B}O_n(q)$ are finitely generated groups whose torsion consists entirely of elements of order 2 or possibly 4.

These results can be sharpened in many cases to give a complete description of our groups. For example

THEOREM 2. $f_\psi$ gives an isomorphism between $\mathcal{B}n_n(q)$ and the subgroup of $\mathcal{R}_n(BO(q))$ consisting of all elements $[M, \xi]$ which satisfy condition (V) above.

For a determination of the plane field bordism groups with other orientedness assumptions see also [6] for $q=1$ and [7] for $q=2$.

If we also take vanishing conditions for the Pontrjagin numbers into account we may in many cases avoid the restriction $n>2q-2$. This can be done either by also considering singularities with higher dimensional kernel, or by applying our approach to complementary $(n-q)$-plane fields. Thus, e.g., Corollary 1 and Theorem 2 turn out to hold whenever $0\leq q \leq n$, the latter as a consequence of the following duality result.

THEOREM 3. If $0\leq q \leq n$, there is a natural isomorphism $\mathcal{B}n_n(q) \cong \mathcal{B}n_n(n-q)$ obtained by taking complements.

This is not a priori obvious since the standard bordism relation for $q$-plane fields induces a different (stabilized) bordism relation for the complementary $(n-q)$-plane fields.

Next define $\mathcal{B}n_n(q)$, $\mathcal{B}n_n(q)$, $\mathcal{B}O_n(q)$ and $\mathcal{B}O_n(q)$ to be the bordism groups of closed $n$-manifolds with smooth $q$-codimensional foliations, satisfying the indicated (co)-orientedness conditions. For $q \geq 2$ Thurston [13] has shown recently that a foliation on a compact manifold $M$ is essentially given by an $(S)\Gamma$-structure $\gamma$ on $M$ (in the sense of Haefliger [3]) together with a bundle monomorphism from the normal bundle $v(\gamma)$ into $TM$.

Thus when we compare our foliation bordism groups with the corresponding usual bordism groups of Haefliger's classifying spaces $B\Gamma(q)$ and $B\Sigma(q)$, we are only confronted with a plane field problem and can apply our approach. We obtain for the forgetful homomorphism $f_\psi: \mathcal{B}n_n(\psi)(q) \to \mathcal{R}_n(B(S)\Gamma(q))$, resp. $f_\psi: \mathcal{B}O_n(\psi)(q) \to \mathcal{R}_n(B(S)\Sigma(q))$:

THEOREM 1'. Let $q \geq 2$ and $n>2q-2$. Then under all four orientedness assumptions, an element $[M, \gamma]$ of the $n$-dimensional bordism group of $B(S)\Gamma(q)$ lies in the image of $f_\psi$ if and only if the vanishing condition (V) is satisfied by the normal bundle $\xi=v(\gamma)$. Furthermore the kernel as well as the cokernel of $f_\psi$ are finite groups consisting entirely of elements of order 2.

\*\* ADDED IN PROOF. More recent work of Thurston implies that the results of this paper still hold for foliations of codimension $q=1$.\*\*
This contrasts with the fact that the foliation bordism groups themselves need not even be countably generated. E.g., \( \mathcal{F} \Omega_{2q+1}(q) \) surjects onto \( R \) for even positive \( q \) (see [14]).

**Theorem 2'.** If \( q \geq 2 \) and \( n \geq 2q-2 \), then \( f^* \) gives an isomorphism between \( \Psi \mathcal{N}(n,q) \) and the subgroup of \( \mathcal{N}(B\Gamma(q)) \) consisting of all elements \([M, \gamma]\) for which the normal bundle \( \xi = n(\gamma) \) satisfies condition (V).

As a corollary to the proof we have

**Theorem 4.** For \( q \geq 1 \), \( n \geq 2q-2 \), every \( q \)-plane field on a closed \( n \)-manifold is bordant (in \( \Psi \mathcal{N}(n,q) \)) to one which is transversal to a foliation of co-dimension \( q \).

The case \( q = 1 \) (where Thurston's results are not available\(^8\)) was settled in [8] by an explicit construction of enough foliations to generate \( \Psi \mathcal{N}(1) \) by their normal linefields.

Finally, note that the singularity approach can also be fruitfully applied to the bordism of manifolds with tangent \( \Gamma \)-frames, or to the bordism of immersions and, more generally, of \( k \)-mersions. More details on this point will appear elsewhere (see also [9]).

I would like to thank Peter Landweber for many helpful references.

2. **The singularity isomorphism.** Let \( \mathcal{N}(BO(q), \Psi) \) (resp. \( \mathcal{N}(B\Gamma(q), \Psi) \)) be the bordism group of triples \((M, \xi, h')\) (resp. \((M, \gamma, h')\)) where \( M \) is a compact smooth \( n \)-manifold, \( \xi \) is a \( q \)-plane bundle over \( M \) (resp. \( \gamma \) is a \( \Gamma(q) \)-structure on \( M \), and we write \( \xi \) for its normal bundle \( n(\gamma) \)), and \( h':\xi|\delta M \rightarrow T(\delta M) \) is a bundle monomorphism. Denote the normal bundle map from \( \Psi \mathcal{N}(B\Gamma(q), \Psi) \) into \( \Psi \mathcal{N}(BO(q), \Psi) \) by \( r^* \).

Now for \( 0 \leq p \leq q \) consider the \( p \cdot (n-q+p) \)-codimensional submanifold \( A_p \) of the total space of the homomorphism bundle \( \text{Hom}(\xi, TM) \) where \( A_p = \bigcup_{x \in M} A_p(x) \) and \( A_p(x) = \{ g: \xi_x \rightarrow T_xM | g \text{ linear, } \dim(\ker(g)) = p \} \) (cf. [5, p. 120]). If \( n \geq 2q-3 \), or equivalently, if \( 2(n-q+2) > n \), then, by transversality we can extend \( h' \) to a vector bundle morphism \( h:\xi \rightarrow TM \) which, as a section in \( \text{Hom}(\xi, TM) \), goes entirely into \( A_0 \cup A_1 \) and intersects \( A_1 \) transversally. Denote by \( S \) the closed \((q-1)\)-dimensional submanifold \( h^{-1}(A_1) \) of the interior of \( M \). Since \( h|S \) has constant rank, there are canonical vector bundles \( \text{Ker}, \text{Coker}, \) and \( \text{Im} \) over \( S \) of dimension 1, \( n-q+1 \), and \( q-1 \), respectively, where e.g., the fiber of \( \text{Ker} \) at \( x \in S \) is the kernel of \( h_x: \xi_x \rightarrow T_xM \). These bundles are related to \( \xi|S \), \( TM|S \) and the

---

\(^8\) See footnote 2.
normal bundle \( v(S, M) \) of \( S \) in \( M \) by the following isomorphisms (which are canonical up to homotopy)

\[
\begin{align*}
\xi &| S \cong \text{Im} \oplus \text{Ker}, \\
TM &| S \cong \text{Im} \oplus \text{Coker}, \\
v(S, M) &\cong \text{Hom} \text{(Ker, Coker)};
\end{align*}
\]

and consequently

\[
(2) \quad i: \text{Im} \oplus \text{Coker} \cong TS \oplus \text{Hom} \text{(Ker, Coker)}.
\]

Associating the bordism class of \((S, \text{Ker, Coker})\) to the class of 
\((M, \xi, h')\), we obtain a well-defined homomorphism

\[
\sigma: \mathcal{N}_n(BO(q), \Psi) \to \mathcal{N}_{q-1}(BO(1) \times BO(n - q + 1)) \\
\cong \mathcal{N}_{q-1}(BO(1) \times BO(q)),
\]

provided \( n \geq 2q - 2 \). Similarly \( \sigma \) is defined on the relative bordism groups 
\( \mathcal{N}_n(BO(q), \Psi) \) and \( \Omega_n(B(S)O(q), \Psi) \) corresponding to the other orientation cases.

We will say that an element \( x = [S, \xi, \eta] \) of \( \mathcal{N}_{q-1}(BO(1) \times BO(q)) \) satisfies condition \( O_b \) (resp. \( O_m \)) for \((n, q)\) if all those Whitney numbers vanish which either involve \( w_1(S) + (n-q)w_1(\xi) \) as a factor or which are made up entirely by a positive number of factors of the form \( n \cdot w_{2k}(S)^2 \) or \( n \cdot w_{2k}(\eta)^2 \), \( k \geq 0 \) (resp. if all Whitney numbers of \( x \) involving \( w_1(S) + (n-q)w_1(\xi) + w_1(\eta) \) vanish).

**THEOREM 5.** Let \( n > 2q - 2 \). Then under all four orientedness assumptions \( \sigma \) is an isomorphism into \( \mathcal{N}_{q-1}(BO(1) \times BO(q)) \). An element \( x \) of

\[
\mathcal{N}_{q-1}(BO(1) \times BO(q))
\]

lies in the image of \( \mathcal{N}_n(BO(q), \Psi) \) (resp. \( \mathcal{N}_n(BO(q), \Psi) \), resp. \( \Omega_n(BO(q), \Psi) \), resp. \( \Omega_n(BO(q), \Psi) \)) under \( \sigma \) if and only if \( x \) is arbitrary (resp. \( x \) satisfies condition \( O_b \), resp. \( O_m \), resp. \( O_b \) and \( O_m \), for \((n, q)\)).

If in addition \( q \geq 2 \), then in all four orientedness cases \( \sigma \circ v_* \) is also an isomorphism onto the image of \( \sigma \).

In particular, for fixed \( q \) the relative bordism groups of a given orientation type depend only on the parity of \( n \).

In the proof we use generalized surgery with core manifolds of dimension \( q \) or 1 or 2. The construction extends to the case of \( \Gamma \)-structures since the normal bundle map \( v: B\Gamma(q) \to BO(q) \) has a \( q \)-connected homotopic fiber [3].

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The relevance of Theorem 5 stems from the following commutative
diagram and its analogues in the other orientation cases
\[ \cdots \rightarrow \mathcal{Y}_n(q) \xrightarrow{j} \mathcal{N}_n(\mathcal{B}(q)) \xrightarrow{j} \mathcal{N}_n(\mathcal{B}(q), \mathcal{Y}) \xrightarrow{\partial} \mathcal{Y}_n-1(q) \xrightarrow{\partial} \mathcal{Y}_n-1(q) \rightarrow \cdots \]
\[ \cdots \rightarrow \mathcal{Y}_n(q) \xrightarrow{j} \mathcal{N}_n(\mathcal{B}(q)) \xrightarrow{j} \mathcal{N}_n(\mathcal{B}(q), \mathcal{Y}) \xrightarrow{\partial} \mathcal{Y}_n-1(q) \xrightarrow{\partial} \mathcal{Y}_n-1(q) \rightarrow \cdots \]
\[ (3) \]
Here the forgetful homomorphisms \( j \) and \( \partial \) make the horizontal sequences
exact.

In order to describe \( \sigma \circ j \) in terms of Whitney numbers, assume \( M \) to be
closed in the discussion above. In a Whitney number of \( (S, \text{Ker}, \text{Coker}) \)
eliminate first \( w(S) \), and then \( w(\text{Coker}) \), using (1) and (2), and apply the
identity
\[ w_1(\text{Ker})^\bullet \cdot (w(TM - \xi)^\alpha w(M)^\beta | S)[S] = w_{n-q+1+\alpha}(TM - \xi)w(TM - \xi)^\alpha w(TM)^\beta [M], \]
where \( \alpha, \beta \) are multi-indices.

Now Theorem 5 implies Theorem 1, Corollary 1 and Theorem 1'. To
obtain a full description of the bordism groups of \( q \)-plane fields, it remains
only to determine the image of \( j \), or equivalently, of \( \sigma \circ j \) (and to check for
possible 4-torsion in \( \Psi \Omega_{n+1}(q) \)). For example, a geometric construction
yields

**THEOREM 6.** For \( n \geq 2q-2 \), the homomorphism \( \sigma \circ j : \mathcal{N}_n(BO(q)) \rightarrow \mathcal{N}_{q-1}(BO(1) \times BO(q)) \) is onto.

Thus, if no orientation conditions are imposed, the lower horizontal
line in diagram (3) breaks down into short exact sequences \( (\partial = 0) \), and so
does the upper line since the middle homomorphism \( \varepsilon_* \) is surjective here
(compare [1]). This proves Theorems 2 and 2'. Theorem 4, or equivalently,
the surjectivity of the left hand homomorphism \( v_* \), follows immediately.

**REFERENCES**

und ihrer Grenzgebiete, N.F., Band 33, Academic Press, New York; Springer-Verlag,
Berlin, 1964. MR 31 #750.
Nuffic Summer School), Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin, 1971,
pp. 133–163. MR 44 #2251.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use


\textsc{Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903}