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ON SEQUENCES OF MEASURES

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Dieudonné [2] has shown that a sequence (μ_n) of regular Borel measures on a compact space X converges weakly, i.e., on all bounded Borel functions, if only it converges on all open Baire sets. The result continues to hold if the μ_n are weakly compact linear maps from $C(X)$ to a locally convex vector space F. Such maps have an integral extension to all bounded Borel functions ϕ, and $\int \phi \, d\mu_n$ converges provided $\int_O d\mu_n$ converges for all open sets O [4], [5]. The Vitali-Hahn-Saks theorem is the set-function analogue of these results.

In this note the analogue of these results for sequences (μ_n) of measures with values in an arbitrary topological vector space F will be proved. In order to deal with set functions and linear maps at the same time, we work in the setting of Daniell-Stone, and consider linear maps $\mu:\mathscr{R}\rightarrow F$, where $\mathscr{R}$ is a vector lattice of real-valued functions on a set X closed under the Stone-operation $\phi\rightarrow\phi\wedge 1$, an "integration lattice" [1]. The examples we have in mind are (1) $\mathscr{R}=C^{00}(X)$, where X is locally compact, (2) $\mathscr{R}=\mathscr{E}(\mathscr{C})$, the step functions over a clan of sets on X, (3) $\mathscr{R}=c^{00}$, (4) $\mathscr{R}=l^{\infty}$. If an additive set function $\mu:\mathscr{C}\rightarrow F$ on the clan $\mathscr{C}$ is given, we extend it by linearity to $\mathscr{E}(\mathscr{C})$ and are in the present situation.

We denote by $\mathscr{O}_0^S$ the collection of sets in X whose indicator is majorized by a function in $\mathscr{R}$ and is the supremum of a sequence in $\mathscr{R}_+$. $\mathscr{O}_0^S$ consists of the open dominated $\mathscr{R}$-Baire sets [1]. We shall assume that every function in $\mathscr{R}$ is bounded and vanishes off some set in $\mathscr{O}_0^S$. Examples (1)–(4) have this property.

Then $\mathscr{R}$ is the union of the normed spaces $\mathscr{R}[O]=\{\phi\in\mathscr{R}:\phi=0 \text{ off } O\}$ under the supremum norm $\| \ \|_\infty$ and is given the inductive limit topology. X is given the initial uniformity and topology for the functions $\phi:X\rightarrow \bar{\boldsymbol{R}}$ $(\phi\in\mathscr{R})$, under which it is precompact. Its completion $\tilde{X}$ can be identified with the set of all Riesz-space characters $t:\mathscr{R}\rightarrow\boldsymbol{R}$ having $t(\phi\wedge 1)= t(\phi)\wedge 1$. Subtracting from $\tilde{X}$ the zero character, one obtains the locally compact *spectrum* $\hat{X}$ of $\mathscr{R}$. X is dense in $\hat{X}$, and the extensions $\hat{\phi}$ of $\phi\in\mathscr{R}$ to $\hat{X}$, the Gelfand transforms, are dense in $C^{00}(\hat{X})$. For the details see [1].

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A function is called $\mathscr{R}$-Baire if it belongs to the smallest family containing $\mathscr{R}$ and closed under pointwise limits of sequences. The vector lattice of bounded $\mathscr{R}$-Baire functions vanishing off some set of $\mathscr{O}_0^S$ is denoted by $\mathscr{R}^S$.

A linear map $\mu:\mathscr{R}\rightarrow F$ will be called extendible[1] if there is an extension $\int\cdot d\mu:\mathscr{R}^S\rightarrow\tilde{F}$ [2] satisfying Lebesgue's dominated convergence theorem. For this to be the case it is evidently necessary that

(C) $\mu:\mathscr{R}\rightarrow F$ is continuous,

(S) for every sequence (ϕ_n) in $\mathscr{R}_+$ decreasing pointwise to zero, $\mu(\phi_n)\rightarrow 0$ in F, and

(G) for every sequence (ϕ_n) in $\mathscr{R}_+$ such that $\sum_{n=1}^{\infty}\phi_n\in\mathscr{R}^S$, $\mu(\phi_n)\rightarrow 0$ in F.[3]

If μ is the extension by linearity of a set function μ_0 on a clan, then (C) signifies that μ_0 has finite semivariation. (S) is automatically satisfied if X is locally compact in the $\mathscr{R}$-topology, by Dini's theorem. When F is locally convex then (G) is equivalent to μ being weakly compact, as Grothendieck has shown **[4]**, **[5]**. Given (C), (G) is evidently automatically satisfied when F is a C-space, i.e., any sequence in F, all of whose finite partial sums form a bounded set, necessarily converges to zero.

(C), (S), and (G) together are also sufficient for the extendability of μ. To see this, let D be a fundamental system of translation-invariant pseudometrics defining the topology of F. Let $\mathscr{R}_\uparrow^S$ denote the suprema of sequences in $\mathscr{R}_+$. For $d\in D$ and $h\in\mathscr{R}_\uparrow^S$ define

$$\mu_d^*(h)=\sup\{d(\mu(\phi)):h\geqq\phi\in\mathscr{R}_+\},$$

and for an arbitrary $f:X\rightarrow\bar{\boldsymbol{R}}_+$ let

$$\mu_d^*(f)=\inf\{\mu_d^*(h):f\leqq h\in\mathscr{R}_\uparrow^S\}.$$

One checks easily (but slightly laboriously) just as in **[1]**, **[3]**, **[5]** that μ_d^* has all the defining properties of a weak upper gauge **[1]** except positive-homogeneity. The latter is replaced by $\mu_d^*(\lambda\phi)\rightarrow 0$ as $\lambda\downarrow 0$ for each $\phi\in\mathscr{R}_+$. Routine arguments then show that the closure of $\mathscr{R}$ in $\boldsymbol{R}^X$, $\mathscr{L}^1(\mathscr{R},\mu_d^*)$, is a complete space under the pseudometric $f\rightarrow\mu_d^*(|f|)$ in which pointwise a.e. convergent and majorized sequences converge in mean, and which therefore contains $\mathscr{R}^S$. Therefore

$$\mathscr{R}^S\subset\mathscr{L}^1(\mathscr{R},\mu)=\bigcap_{d\in D}\mathscr{L}^1(\mathscr{R},\mu_d^*)$$

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[1] Cf. **[3]**.

[2] $\tilde{F}$ denotes the completion of F.

[3] It is sufficient to require (G) only for sequences (ϕ_n) in $\mathscr{R}_+$ with sum in $\mathscr{R}^S$ and with mutually disjoint carriers $[\phi_n>0]$.

and μ has an extension, continuous with respect to the collection of translation invariant pseudometrics μ_d^*, $d \in D$, from all of $\mathscr{L}^1(\mathscr{R}, \mu)$ to $\tilde{F}$ (see footnote 2).

Following is the main result. In it $\mathscr{H}$ denotes the set of all bounded functions $h: X \to \boldsymbol{R}_+$ whose carrier $[h>0]$ belongs to $\mathscr{O}_0^S$ and that are continuous on $[h>0]$.

THEOREM. *Let (μ_n) be a sequence of extendible maps from $\mathscr{R}$ to F. If $\lim_{n\to\infty} \int h\, d\mu_n$ exists in F for all $h \in \mathscr{H}$, then $\mu_\infty(\phi) = \lim_{n\to\infty} \mu_n(\phi)$ $(\phi \in \mathscr{R})$ defines an extendible measure $\mu_\infty: \mathscr{R} \to F$, and $\int f\, d\mu_\infty = \lim_{n\to\infty} \int f\, d\mu_n$ for all $f \in \mathscr{R}^S$.*

To prove this, we shall consider below the map $U: \mathscr{R} \to c_F$ into the space c_F of convergent sequences in F that is given by $U(\phi)(k) = \mu_k(\phi)$. U is evidently extendible if c_F is given the topology p of pointwise convergence. The proof of the Theorem will consist essentially in showing that U is extendible if c_F is given the topology u of uniform convergence. A major step will be to prove that p has the Orlicz property for u.

If $\sigma \subset \tau$ are two linear Hausdorff topologies on a vector space E then σ is said to have the *Orlicz property* for τ provided every sequence (ξ_n) in E, all of whose subsequences are σ-summable to an element of E, necessarily τ-converges to zero. If (F, τ) is complete then such a sequence (and all of its subsequences) is actually τ-summable; indeed, for any increasing sequence $(n(k))$, $\xi_k' = \sum_{i=n(k)}^{n(k+1)} \xi_i$ is a sequence, all of whose subsequences are σ-summable in E, and hence $\tau - \lim_{k\to\infty} \xi_j' = 0$. By Cauchy's criterion, (ξ_n) is summable in (F, τ).

PROPOSITION. *Let F be a Hausdorff topological vector space, and denote by c_F the space of convergent sequences in F. The topology p of pointwise convergence has the Orlicz property for the topology u of uniform convergence on c_F.*

PROOF. Let (f_n) be a sequence in c_F all of whose subsequences are p-summable to an element of c_F. We have to show that, for every continuous translation-invariant pseudometric d on F,

$$d_\infty(f_n) = \sup_{k \in \boldsymbol{N}} d(f_n(k), 0)$$

converges to zero as $n \to \infty$. Viewing (f_n) as a sequence in the Hausdorff completion of the pseudometric space (F, d), we may assume that F is actually complete and metrizable with translation-invariant metric d.

For each $n \in \boldsymbol{N}$ set $f_n(\infty) = \lim_{k\to\infty} f_n(k)$. We show first that $f_n(\infty) \to 0$ as $n \to \infty$. We proceed by contradiction and, extracting a subsequence, assume that $d(f_n(\infty)) > c$ for all $n \in \boldsymbol{N}$ and some $c > 0$. Given an $\varepsilon > 0$, we define inductively two increasing sequences $(K(i))$ and $(N(i))$ in $\boldsymbol{N}$ such that

$$(*)\qquad \begin{aligned} d(f_n(k)) &\leqq \varepsilon 2^{-i} \quad \text{for } k \leqq K(i) \text{ and } n \geqq N(i), \\ d(f_{N(i)}(k) - f_{N(i)}(\infty)) &< \varepsilon 2^{-i} \quad \text{for } k \geqq K(i+1), \end{aligned}$$

which is possible since $f_n(k) \to 0$ as $n \to \infty$ for each $k \in \boldsymbol{N}$. By assumption, the pointwise sum $f = \sum_{i=1}^{\infty} f_{N(i)}$ belongs to c_F and has a limit $f(\infty) = \lim_{k\to\infty} f(k)$. Now,

$$d\left(f(\infty) - \sum_{i=1}^{j} f_{N(i)}(\infty)\right)$$
$$\leqq d(f(\infty) - f(K(j))) + \sum_{i=j}^{\infty} d(f_{N(i)}(K(j))) + \sum_{i=1}^{j-1} d(f_{N(i)}(K(j)) - f_{N(i)}(\infty)).$$

The first term on the right can be made smaller than ε by the choice of j, and the two remaining terms are smaller than ε each by (*). Hence $d(f_{N(i)}(\infty)) \to 0$ as $i \to \infty$, after all. By the condensation argument above, $(f_n(\infty))$ is actually summable in the completion of F, and so are all of its subsequences. Replacing f_n by $f_n - f_n(\infty)$, we may therefore assume that all the f_n belong to the space c_F^0 of nullsequences.

To show that $f_n \to 0$ uniformly, we proceed by contradiction and, extracting a subsequence if necessary, assume that $d_\infty(f_n) > c$ for all $n \in \boldsymbol{N}$ and some $c > 0$.

We define again sequences $(N(i))$ and $(K(i))$ satisfying (*) (with $f_n(\infty) = 0$ for all $n \in \boldsymbol{N}$) and set

$$\begin{aligned} f'_{N(i)}(k) &= f_{N(i)}(k) \quad \text{for } K(i) < k < K(i+1), \\ &= 0 \qquad\qquad \text{for all other } k. \end{aligned}$$

Then $d_\infty(f'_{N(i)}, f_{N(i)}) < \varepsilon 2^{-i}$ for all $i \in \boldsymbol{N}$, and consequently $(f'_{N(i)})$ is pointwise summable to an element of c_F^0. Indeed, we have

$$\sum_{i=1}^{\infty} f'_{N(i)} = \sum_{i=1}^{\infty} f_{N(i)} + \sum_{i=1}^{\infty} (f_{N(i)} - f'_{N(i)}) \in c_F^0$$

in the pointwise topology. (Note that $\sum_{i=1}^{\infty} (f_{N(i)} - f'_{N(i)})$ exists in the uniform topology of the complete space c_F^0.) From the fact that the $f'_{N(i)}$ have mutually disjoint carriers, it is obvious that $d_\infty(f'_{N(i)}) \to 0$ as $i \to \infty$. Hence $c_\infty(f'_{N(i)}) \to 0$ as $i \to \infty$, after all.

We are now ready to prove the Theorem. This is done by showing that the map $U: \mathscr{R} \to c_F$ satisfies (C), (S), and (G) and thus is extendible; the statements of the Theorem are then evidently true.

For (C), it suffices to prove the continuity of the restrictions of U to $\mathscr{R}[O]$, $O \in \mathscr{O}_0^S$. If one of them is not, then there are $\phi_n \in \mathscr{R}[O]$ with

$\|\phi_n\|_\infty \leqq 2^{-n}$ and $d_\infty(U(\phi_n))>c$ for some $d\in D$ and some $c>0$. This is absurd, though, since $(U(\phi_n))$ is a sequence in c_F all of whose subsequences are p-summable to an element of c_F, whence a contradiction to the proposition.

The proof of (G) is similar. Let (ϕ_n) be a sequence in $\mathscr{R}_+$ with disjoint carriers $[\phi_n>0]$ (see footnote 3) and sum in $\mathscr{R}^S$. Then for any subset A of N, $\sum_{n\in A}\phi_n\in\mathscr{H}$, and $\sum_{n\in A}U(\phi_n)=\int\sum_{n\in A}\phi_n\,dU\in c_F$ exists in the pointwise topology of c_F. By the Proposition, $\lim_{n\to\infty}U(\phi_n)=0$ in the uniform topology of c_F.

It remains to prove (S). Let (ϕ_n) be a decreasing sequence in $\mathscr{R}_+$ with pointwise limit zero. We consider the Gelfand-Bauer transform $\hat{U}:\hat{\mathscr{R}}\to c_F$, defined for every Gelfand transform $\hat{\phi}$ of an element $\phi\in\mathscr{R}$ by $\hat{U}(\hat{\phi})=U(\phi)$. From Dini's theorem and the local compactness of $\tilde{X}$, $\hat{U}$ satisfies (S). Since it evidently satisfies (C) and (G) as well, it is extendible.

Let $g=\inf_{n\in N}\hat{\phi}_n$. Then g is an upper semicontinuous Baire function of compact support on $\hat{X}$, and by the dominated convergence theorem

$$\int g\,d\hat{U}=\lim_{n\to\infty}\hat{U}(\hat{\phi}_n)=\lim_{n\to\infty}U(\phi_n)$$

exists in $c_{\tilde{F}}$. For any $k\in N$, we have

$$\left(\int g\,d\hat{U}\right)(k)=\lim_{n\to\infty}U(\phi_n)(k)=\lim_{n\to\infty}\mu_k(\phi_n)=0,$$

and so $\lim_{n\to\infty}U(\phi_n)=0$, as claimed.

REMARKS. (1) The proof shows that the $\mu_1,\cdots,\mu_\infty$ are actually uniformly extendible in the sense that if a majorized sequence (f_n) in $\mathscr{R}^S$ (or in $\mathscr{L}^1(\mathscr{R},U)$) converges pointwise to some f, then $\int f_n\,d\mu_k\to\int f\,d\mu_k$ in $\tilde{F}$ uniformly in $k=1,\cdots,\infty$; indeed, we have $\int f_n\,dU\to\int f\,dU$ in $c_{\tilde{F}}$.

(2) If F is locally convex, it suffices to require that $\int_O d\mu_k$ converges in F for all $O\in\mathscr{O}_0^S$, and the same conclusion holds. The proof of this by Thomas **[5]**[4] for the case that X is locally compact in the $\mathscr{R}$-topology can be easily adapted to our setting using the Gelfand-Bauer transform. Turning then to the special case where $\mathscr{R}$ is the step functions over a clan $\mathscr{C}$, one obtains the following result: If (μ_k) is a sequence of σ-additive F-valued set functions of finite semivariation, then $\int f\,d\mu_k\to\int f\,d\mu_\infty$ for all $f\in\mathscr{R}^S$ and some σ-additive set function μ_∞ provided $\lim_{k\to\infty}\int_O d\mu_k$ exist in F for every set O that is a subset of a set of $\mathscr{C}$ and is the countable union of sets in $\mathscr{C}$ (when F is locally convex), or provided that $\lim_{k\to\infty}\int\phi\,d\mu_k$ exist in F for every bounded function ϕ that vanishes off a set of $\mathscr{C}$ and is a countable linear combination of indicators of sets in $\mathscr{C}$.

[4] Our proof uses essentially Thomas' technique.

(3) Let E be a Banach space, and let $\mathscr{R}\otimes E$ denote the collection of functions $x \to \sum \phi_i(x)\xi_i$ ($\phi_i \in \mathscr{R}$, $\xi_i \in E$, the sum finite), equipped with the obvious inductive limit topology [**1**]. The arguments given above can be adapted to prove the following. Let $\mu_k : \mathscr{R}\otimes E \to F$ be a sequence of extendible maps such that $\int h\, d\mu_k$ converges in F for each bounded $\mathscr{R}$-Baire function $h: X \to E$ such that $[h \neq 0] \in \mathscr{O}_0^S$, and such that h is continuous on $[h \neq 0]$. Then there exists an extendible map $\mu_\infty : \mathscr{R}\otimes E \to F$ such that $\int f\, d\mu_k \to \int f\, d\mu_\infty$ for all bounded E-valued $\mathscr{R}$-Baire functions vanishing off some set of $\mathscr{O}_0^S$, and $\mu_1, \cdots, \mu_\infty$ is uniformly extendible.[5]

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[5] For the terminology, see [**1**].