Dieudonné [2] has shown that a sequence \((\mu_n)\) of regular Borel measures on a compact space \(X\) converges weakly, i.e., on all bounded Borel functions, if only it converges on all open Baire sets. The result continues to hold if the \(\mu_n\) are weakly compact linear maps from \(C(X)\) to a locally convex vector space \(F\). Such maps have an integral extension to all bounded Borel functions \(\phi\), and \(\int \phi \, d\mu_n\) converges provided \(\int_O d\mu_n\) converges for all open sets \(O\) [4], [5]. The Vitali-Hahn-Saks theorem is the set-function analogue of these results.

In this note the analogue of these results for sequences \((\mu_n)\) of measures with values in an arbitrary topological vector space \(F\) will be proved. In order to deal with set functions and linear maps at the same time, we work in the setting of Daniell-Stone, and consider linear maps \(\mu : \mathcal{R} \to F\), where \(\mathcal{R}\) is a vector lattice of real-valued functions on a set \(X\) closed under the Stone-operation \(\phi \mapsto \phi \wedge 1\), an "integration lattice" [1]. The examples we have in mind are (1) \(\mathcal{R} = C^0(X)\), where \(X\) is locally compact, (2) \(\mathcal{R} = C^0(X)\), the step functions over a clan of sets on \(X\), (3) \(\mathcal{R} = C^0\), (4) \(\mathcal{R} = L^\infty\). If an additive set function \(\mu : \mathcal{C} \to F\) on the clan \(\mathcal{C}\) is given, we extend it by linearity to \(\mathcal{C}(\mathcal{C})\) and are in the present situation.

We denote by \(\mathcal{O}^S_\mathcal{R}\) the collection of sets in \(X\) whose indicator is majorized by a function in \(\mathcal{R}\) and is the supremum of a sequence in \(\mathcal{O}_{\mathcal{R}}\). \(\mathcal{O}^S_\mathcal{R}\) consists of the open dominated \(\mathcal{R}\)-Baire sets [1]. We shall assume that every function in \(\mathcal{R}\) is bounded and vanishes off some set in \(\mathcal{O}^S_\mathcal{R}\). Examples (1)–(4) have this property.

Then \(\mathcal{R}\) is the union of the normed spaces \(\mathcal{R}[O] = \{\phi \in \mathcal{R} : \phi = 0 \text{ off } O\}\) under the supremum norm \(\|\cdot\|_\infty\) and is given the inductive limit topology. \(X\) is given the initial uniformity and topology for the functions \(\phi : X \to \mathcal{R}\) (\(\phi \in \mathcal{R}\)), under which it is precompact. Its completion \(\hat{X}\) can be identified with the set of all Riesz-space characters \(t : \mathcal{R} \to \mathcal{R}\) having \(t(\phi \wedge 1) = t(\phi)\wedge 1\). Subtracting from \(\hat{X}\) the zero character, one obtains the locally compact spectrum \(\hat{X}\) of \(\mathcal{R}\). \(X\) is dense in \(\hat{X}\), and the extensions \(\hat{\phi}\) of \(\phi \in \mathcal{R}\) to \(\hat{X}\), the Gelfand transforms, are dense in \(C^0(\hat{X})\). For the details see [1].
A function is called \( R \)-Baire if it belongs to the smallest family containing \( R \) and closed under pointwise limits of sequences. The vector lattice of bounded \( R \)-Baire functions vanishing off some set of \( C_0 \) is denoted by \( R^S \).

A linear map \( \mu: R \to F \) will be called extendible\(^1\) if there is an extension \( \int d\mu: R^S \to F^2 \) satisfying Lebesgue's dominated convergence theorem. For this to be the case it is evidently necessary that

(C) \( \mu: R \to F \) is continuous,

(S) for every sequence \( (\phi_n) \) in \( R^+ \) decreasing pointwise to zero, \( \mu(\phi_n) \to 0 \) in \( F \), and

(G) for every sequence \( (\phi_n) \) in \( R^+ \) such that \( \sum_{n=1}^{\infty} \phi_n \in R^S \), \( \mu(\phi_n) \to 0 \) in \( F \).\(^3\)

If \( \mu \) is the extension by linearity of a set function \( \mu_0 \) on a clan, then (C) signifies that \( \mu_0 \) has finite semivariation. (S) is automatically satisfied if \( X \) is locally compact in the \( R \)-topology, by Dini's theorem. When \( F \) is locally convex then (G) is equivalent to \( \mu \) being weakly compact, as Grothendieck has shown [4], [5]. Given (C), (G) is evidently automatically satisfied when \( F \) is a C-space, i.e., any sequence in \( F \), all of whose finite partial sums form a bounded set, necessarily converges to zero.

(C), (S), and (G) together are also sufficient for the extendability of \( \mu \).

To see this, let \( D \) be a fundamental system of translation-invariant pseudometrics defining the topology of \( F \). Let \( R^S_1 \) denote the suprema of sequences in \( R^+ \). For \( d \in D \) and \( h \in R^S_1 \) define

\[
\mu^*_d(h) = \sup\{d(\mu(\phi)): h \geq \phi \in R^+_1\},
\]

and for an arbitrary \( f: X \to R^+_1 \) let

\[
\mu^*_d(f) = \inf\{\mu^*_d(h): f \leq h \in R^S_1\}.
\]

One checks easily (but slightly laboriously) just as in [1], [3], [5] that \( \mu^*_d \) has all the defining properties of a weak upper gauge [1] except positive-homogeneity. The latter is replaced by \( \mu^*_d(\lambda \phi) \to 0 \) as \( \lambda \downarrow 0 \) for each \( \phi \in R^+_1 \).

Routine arguments then show that the closure of \( R \) in \( R^X \), \( L^1(R, \mu^*_d) \), is a complete space under the pseudometric \( d \to \mu^*_d([f]) \) in which pointwise a.e. convergent and majorized sequences converge in mean, and which therefore contains \( R^S \). Therefore

\[
R^S \subset L^1(R, \mu) = \bigcap_{d \in D} L^1(R, \mu^*_d)
\]

---

\(^1\) Cf. [3].

\(^2\) \( F \) denotes the completion of \( F \).

\(^3\) It is sufficient to require (G) only for sequences \( (\phi_n) \) in \( R^+ \), with sum in \( R^S \) and with mutually disjoint carriers \( \{\phi_n\} > 0 \).
and \( \mu \) has an extension, continuous with respect to the collection of translation invariant pseudometrics \( \mu^*_d \), \( d \in \mathcal{D} \), from all of \( L^1(\mathcal{R}, \mu) \) to \( \mathcal{F} \) (see footnote 2).

Following is the main result. In it \( \mathcal{H} \) denotes the set of all bounded functions \( h: X \to \mathbb{R} \) whose carrier \([h>0]\) belongs to \( \mathcal{C}_0^S \) and that are continuous on \([h>0]\).

**Theorem.** Let \( (\mu_n) \) be a sequence of extendible maps from \( \mathcal{H} \) to \( \mathcal{F} \). If \( \lim_{n\to\infty} \int h \, d\mu_n \) exists in \( \mathcal{F} \) for all \( h \in \mathcal{H} \), then \( \mu_n(\phi) = \lim_{n\to\infty} \mu_n(\phi) \) \( (\phi \in \mathcal{R}) \) defines an extendible measure \( \mu_\infty: \mathcal{H} \to \mathcal{F} \), and \( \int f \, d\mu_\infty = \lim_{n\to\infty} \int f \, d\mu_n \) for all \( f \in \mathcal{F}^S \).

To prove this, we shall consider below the map \( U: \mathcal{H} \to \mathcal{C}_F \) into the space \( \mathcal{C}_F \) of convergent sequences in \( \mathcal{F} \) that is given by \( U(\phi)(k) = d(\phi) \). \( U \) is evidently extendible if \( \mathcal{C}_F \) is given the topology \( p \) of pointwise convergence. The proof of the Theorem will consist essentially in showing that \( U \) is extendible if \( \mathcal{C}_F \) is given the topology \( u \) of uniform convergence. A major step will be to prove that \( \mathcal{F} \) has the Orlicz property for \( u \).

If \( \sigma \preceq \tau \) are two linear Hausdorff topologies on a vector space \( E \) then \( \sigma \) is said to have the Orlicz property for \( \tau \) provided every sequence \( (\xi_n) \) in \( E \), all of whose subsequences are \( \sigma \)-summable to an element of \( E \), necessarily \( \tau \)-converges to zero. If \( (F, \tau) \) is complete then such a sequence (and all of its subsequences) is actually \( \tau \)-summable; indeed, for any increasing sequence \( (n(k)) \), \( \xi'_k = \sum_{i=n(k)}^{n(k+1)} \xi_i \) is a sequence, all of whose subsequences are \( \sigma \)-summable in \( E \), and hence \( \tau - \lim_{k \to \infty} \xi'_k = 0 \). By Cauchy’s criterion, \( (\xi_n) \) is summable in \( (F, \tau) \).

**Proposition.** Let \( F \) be a Hausdorff topological vector space, and denote by \( \mathcal{C}_F \) the space of convergent sequences in \( F \). The topology \( p \) of pointwise convergence has the Orlicz property for the topology \( u \) of uniform convergence on \( \mathcal{C}_F \).

**Proof.** Let \( (f_n) \) be a sequence in \( \mathcal{C}_F \) all of whose subsequences are \( p \)-summable to an element of \( \mathcal{C}_F \). We have to show that, for every continuous translation-invariant pseudometric \( d \) on \( F \),

\[
d_\infty(f_n) = \sup_{k \in \mathbb{N}} d(f_n(k), 0)
\]

converges to zero as \( n \to \infty \). Viewing \( (f_n) \) as a sequence in the Hausdorff completion of the pseudometric space \( (F, d) \), we may assume that \( F \) is actually complete and metrizable with translation-invariant metric \( d \).

For each \( n \in \mathbb{N} \) set \( f_n(\infty) = \lim_{k \to \infty} f_n(k) \). We show first that \( f_n(\infty) \to 0 \) as \( n \to \infty \). We proceed by contradiction and, extracting a subsequence, assume that \( d(f_n(\infty)) > c \) for all \( n \in \mathbb{N} \) and some \( c > 0 \). Given an \( \varepsilon > 0 \),
we define inductively two increasing sequences \((K(i))\) and \((N(i))\) in \(N\) such that

\[
d(f_n(k)) \leq \varepsilon 2^{-i} \quad \text{for} \quad k \leq K(i) \text{ and } n \geq N(i),
\]

\((*)\)

\[
d(f_{N(i)}(k) - f_{N(i)}(\infty)) < \varepsilon 2^{-i} \quad \text{for} \quad k \geq K(i + 1),
\]

which is possible since \(f_n(k) \to 0\) as \(n \to \infty\) for each \(k \in N\). By assumption, the pointwise sum \(f = \sum_{i=1}^{\infty} f_{N(i)}\) belongs to \(c_F\) and has a limit \(f(\infty) = \lim_{K \to \infty} f(k)\). Now,

\[
d\left(f(\infty) - \sum_{i=1}^{j} f_{N(i)}(\infty)\right) \leq d(f(\infty) - f(K(j))) + \sum_{i=j+1}^{\infty} d(f_{N(i)}(K(j))) + \sum_{i=1}^{j-1} d(f_{N(i)}(K(j)) - f_{N(i)}(\infty)).
\]

The first term on the right can be made smaller than \(\varepsilon\) by the choice of \(j\), and the two remaining terms are smaller than \(\varepsilon\) each by \((*)\). Hence \(d(f_{N(i)}(\infty)) \to 0\) as \(i \to \infty\), after all. By the condensation argument above, \((f_n(\infty))\) is actually summable in the completion of \(F\), and so are all of its subsequences. Replacing \(f_n\) by \(f_n - f_n(\infty)\), we may therefore assume that all the \(f_n\) belong to the space \(c_F^0\) of nullsequences.

To show that \(f_n \to 0\) uniformly, we proceed by contradiction and, extracting a subsequence if necessary, assume that \(d(\infty,f_n) > \varepsilon\) for all \(n \in N\) and some \(\varepsilon > 0\).

We define again sequences \((N(i))\) and \((K(i))\) satisfying \((*)\) (with \(f_n(\infty) = 0\) for all \(n \in N\)) and set

\[
f'_{N(i)}(k) = f_{N(i)}(k) \quad \text{for} \quad K(i) < k < K(i + 1),
\]

\[
= 0 \quad \text{for all other } k.
\]

Then \(d(\infty,f'_{N(i)},f_{N(i)}) < \varepsilon 2^{-i}\) for all \(i \in N\), and consequently \((f'_{N(i)})\) is pointwise summable to an element of \(c_F^0\). Indeed, we have

\[
\sum_{i=1}^{\infty} f'_{N(i)} = \sum_{i=1}^{\infty} f_{N(i)} + \sum_{i=1}^{\infty} (f_{N(i)} - f'_{N(i)}) \in c_F^0
\]

in the pointwise topology. (Note that \(\sum_{i=1}^{\infty} (f_{N(i)} - f'_{N(i)})\) exists in the uniform topology of the complete space \(c_F^0\).) From the fact that the \(f'_{N(i)}\) have mutually disjoint carriers, it is obvious that \(d(\infty,f'_{N(i)}) \to 0\) as \(i \to \infty\). Hence \(c_\infty(f'_{N(i)}) \to 0\) as \(i \to \infty\), after all.

We are now ready to prove the Theorem. This is done by showing that the map \(U: \mathcal{R} \to c_F\) satisfies \((C)\), \((S)\), and \((G)\) and thus is extendible; the statements of the Theorem are then evidently true.

For \((C)\), it suffices to prove the continuity of the restrictions of \(U\) to \(\mathcal{R}[O]\), \(O \in \mathcal{G}^S_0\). If one of them is not, then there are \(\phi_n \in \mathcal{R}[O]\) with
\[ \| \phi_n \|_\infty \leq 2^{-n} \] and \( d_\omega (U(\phi_n)) > c \) for some \( d \in D \) and some \( c > 0 \). This is absurd, though, since \( (U(\phi_n)) \) is a sequence in \( c_F \) all of whose subsequences are \( p \)-summable to an element of \( c_F \), whence a contradiction to the proposition.

The proof of (G) is similar. Let \( (\phi_n) \) be a sequence in \( \mathcal{B}_+ \) with disjoint carriers \( [\phi_n] > 0 \) (see footnote 3) and sum in \( \mathcal{B}^S \). Then for any subset \( A \) of \( N \), \( \sum_{n \in A} \phi_n \in \mathcal{H} \), and \( \sum_{n \in A} U(\phi_n) = \sum_{n \in A} \phi_n dU \in c_F \) exists in the pointwise topology of \( c_F \). By the Proposition, \( \lim_{n \to \infty} U(\phi_n) = 0 \) in the uniform topology of \( c_F \).

It remains to prove (S). Let \( (\phi_n) \) be a decreasing sequence in \( \mathcal{B}_+ \) with pointwise limit zero. We consider the Gelfand-Bauer transform \( \hat{U} : \mathcal{B} \to c_F \), defined for every Gelfand transform \( \hat{\phi} \) of an element \( \phi \in \mathcal{B} \) by \( \hat{U}(\hat{\phi}) = U(\phi) \). From Dini’s theorem and the local compactness of \( \hat{X} \), \( \hat{U} \) satisfies (S). Since it evidently satisfies (C) and (G) as well, it is extendible.

Let \( g = \inf_{n \in N} \phi_n \). Then \( g \) is an upper semicontinuous Baire function of compact support on \( \hat{X} \), and by the dominated convergence theorem

\[
\int g \, d\hat{U} = \lim_{n \to \infty} \hat{U}(\phi_n) = \lim_{n \to \infty} U(\phi_n)
\]

exists in \( c_F \). For any \( k \in N \), we have

\[
\left( \int g \, d\hat{U} \right)(k) = \lim_{n \to \infty} U(\phi_n)(k) = \lim_{n \to \infty} \mu_k(\phi_n) = 0,
\]

and so \( \lim_{n \to \infty} U(\phi_n) = 0 \), as claimed.

**Remarks.** (1) The proof shows that the \( \mu_1, \cdots, \mu_\infty \) are actually uniformly extendible in the sense that if a majorized sequence \( (f_n) \) in \( \mathcal{B}_\infty \) (or in \( L^1(\mathcal{B}, U) \)) converges pointwise to some \( f \), then \( \int f_n \, d\mu_k \to \int f \, d\mu_k \) in \( F \) uniformly in \( k=1, \cdots, \infty \); indeed, we have \( \int f_n \, d\mu_k \to \int f \, dU \) in \( c_F \).

(2) If \( F \) is locally convex, it suffices to require that \( \int O \, d\mu_k \) converges in \( F \) for all \( O \in \mathcal{O}_0^\infty \), and the same conclusion holds. The proof of this by Thomas [5] for the case that \( X \) is locally compact in the \( \mathcal{B} \)-topology can be easily adapted to our setting using the Gelfand-Bauer transform. Turning then to the special case where \( \mathcal{B} \) is the step functions over a clan \( \mathcal{C} \), one obtains the following result: If \( (\mu_k) \) is a sequence of \( \sigma \)-additive \( F \)-valued set functions of finite semivariation, then \( \int f \, d\mu_k \to \int f \, d\mu_k \) for all \( f \in \mathcal{B}_\infty \) and some \( \sigma \)-additive set function \( \mu_\infty \) provided \( \lim_{k \to \infty} \int O \, d\mu_k \) exist in \( F \) for every set \( O \) that is a subset of a set of \( \mathcal{C} \) and is the countable union of sets in \( \mathcal{C} \) (when \( F \) is locally convex), or provided that \( \lim_{k \to \infty} \int \phi \, d\mu_k \) exist in \( F \) for every bounded function \( \phi \) that vanishes off a set of \( \mathcal{C} \) and is a countable linear combination of indicators of sets in \( \mathcal{C} \).

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4 Our proof uses essentially Thomas’ technique.
Let $E$ be a Banach space, and let $\mathcal{R} \otimes E$ denote the collection of functions $x \mapsto \sum \phi_i(x)\xi_i$ ($\phi_i \in \mathcal{R}$, $\xi_i \in E$, the sum finite), equipped with the obvious inductive limit topology [1]. The arguments given above can be adapted to prove the following. Let $\mu_k : \mathcal{R} \otimes E \rightarrow F$ be a sequence of extendible maps such that $\int h \, d\mu_k$ converges in $F$ for each bounded $\mathcal{R}$-Baire function $h : X \rightarrow E$ such that $[h \neq 0] \in \mathcal{C}_0^S$, and such that $h$ is continuous on $[h \neq 0]$. Then there exists an extendible map $\mu_\infty : \mathcal{R} \otimes E \rightarrow F$ such that $\int f \, d\mu_\infty$ converges for all bounded $E$-valued $\mathcal{R}$-Baire functions vanishing off some set of $\mathcal{C}_0^S$, and $\mu_1, \cdots, \mu_\infty$ is uniformly extendible.

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5 For the terminology, see [1].