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A REPRESENTATION OF CLOSED, ORIENTABLE 3-MANIFOLDS AS 3-FOLD BRANCHED COVERINGS OF S^3

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A *represented link* is a link L in S^3 together with a representation ω of the link group $\pi(S^3-L)$ into the symmetric permutation group of d symbols $\mathscr{S}_d$. Let us call ω *simple* if it represents each meridian of L by an appropriate transposition. If (L, ω) is a represented link, there is a uniquely associated closed, orientable 3-manifold $M(L, \omega)$, namely the d-fold covering of S^3 branched over L, that is determined by the representation ω. It has been proved by J. W. Alexander [**1**] that *every closed orientable 3-manifold is $M(L, \omega)$ for some link L and representation ω.*

H. M. Hilden (personal communication to the author) has proved

THEOREM 1. *Every closed, orientable 3-manifold is $M(K, \omega)$ for some knot K and simple representation ω of $\pi(S^3-K)$ onto $\mathscr{S}_3$.*

Theorem 1 states that every closed, orientable 3-manifold is a 3-fold irregular covering space of S^3 branched over a knot K in such a way that the inverse image of a point of K consists of a point of branch-index 2 and a point of branch-index 1.

We have obtained (independently) a different proof of Theorem 1 which will be sketched here. A detailed proof will appear elsewhere.

Let L be a link in S^3 composed of two unlinked trivial knots P and R, and let ω be a representation of $\pi(S^3-L)$ onto the group $\mathscr{S}_3$ of permutations of the symbols 0, 1, and 2, such that ω represents a meridian of P (resp. R) by the transposition (01) (resp. (02)). It is easy to see that $M(L, \omega)$ is S^3. Let $p: M(L, \omega) \to S^3$ be the covering projection. Then $p^{-1}(P)$ (resp. $p^{-1}(R)$) is composed of a curve P_{01} (resp. R_{02}) of branch-index 2 and a curve P_2 (resp. R_1) of branch-index 1. These curves are unknotted and unlinked. Let B be a ball in S^3 which cuts L in exactly two disjoint arcs in P and such that $p^{-1}(B)$ is the disjoint sum of a ball B_2 which cuts P_2, and a solid torus B_{01} which cuts P_{01}. Then $p|B_{01}: B_{01} \to B$ is

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the branched covering induced by the "symmetry with respect to axis P_{01}". If we remove B from S^3 and resew them differently, in such a way that $\partial B \cap P$ keeps fixed as a set, the effect in the covering is to remove $p^{-1}(B)$ from S^3 and to resew it differently. This is the same as doing surgery on the solid torus B_{01}.

W. B. R. Lickorish has proved [**2**, p. 419] that every closed, orientable 3-manifold can be obtained by surgery on a member of a special family $\mathscr{F}$ of links in S^3. If N' is a link in $\mathscr{F}$, there is a link N, of the same isotopy type as N', such that each component of N cuts $P_{01}+R_{02}$ in exactly two points and is "symmetric" with respect to P_{01} or R_{02}. If N has n components, then there is a union of n disjoint solid balls $B_1, \cdots, B_n$ such that $p^{-1}(\bigcup_{i=1}^n B_i)$ is the disjoint union of a regular neighbourhood of N and n solid balls in $M(L, \omega)$. We can remove the balls $B_1, \cdots, B_n$ from S^3 and resew them differently in order to obtain a link with a simple representation onto $\mathscr{S}_3$, (L', ω'), such that $M(L', \omega')$ is exhibited as a manifold obtained by doing a given surgery on the link N. This shows that each manifold obtained by surgery on a member of $\mathscr{F}$ is $M(L', \omega')$ for some link L' and some simple representation ω' onto $\mathscr{S}_3$. Now, we can apply the modifications defined in [**3**] to (L', ω') to obtain a *knot* with a simple representation onto $\mathscr{S}_3$, (K, ω''), such that $M(K, \omega'')=M(L', \omega')$. This proves Theorem 1.

Note that our proof is constructive in the following sense. If a manifold M is obtained by surgery on a member of $\mathscr{F}$, we can *exhibit* a simple represented knot (K, ω) such that $M(K, \omega)=M$.

References

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