THE ORIENTED TOPOLOGICAL
AND PL COBORDISM RINGS

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1. Introduction and statement of results. In this note we announce results on the 2-local structure of the oriented topological cobordism ring $\Omega_*^{\text{TOP}}$ and its PL analogue $\Omega_*^{\text{PL}}$.

It is a well-known consequence of transversality that

$$\Omega_*^{\text{TOP}} = \pi_*(\text{MSTOP}), \quad * \neq 4 \quad \text{and} \quad \Omega_*^{\text{PL}} = \pi_*(\text{MSPL}),$$

where MSTOP and MSPL are the oriented Thom spectra.

Also, the homotopy theory of these spectra divides into two distinct problems: the theory at the prime 2 and the theory away from 2. We let $\mathbb{Z}(2)$ denote the integers localized at 2 and $\mathbb{Z}[\frac{1}{2}]$ the integers localized away from 2.

Sullivan [9] showed that the free part of $\Omega_*^{\text{TOP}} \otimes \mathbb{Z}[\frac{1}{2}]$ ($=\Omega_*^{\text{PL}} \otimes \mathbb{Z}[\frac{1}{2}]$); $\Omega_*^{\text{TOP}}/\text{Tor} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra with one generator in each dimension congruent to zero mod 4.

At the prime 2 Browder, Liulevicius and Peterson [2] show that the localized spectra MSTOP(2) and MSPL(2) become wedges of Eilenberg-Mac Lane spectra. Hence the homotopy theory is a direct consequence of the homology theory. In particular,

$$\left(\Omega_*^{\text{TOP}}/\text{Tor}\right) \otimes \mathbb{Z}(2) = H_*(\text{BSTOP}; \mathbb{Z}(2))/\text{Tor}$$

and similarly in the PL case.

Let $M_0^{4n}$, $n > 1$, be the Milnor manifold of index 8 constructed by plumbing disk tangent bundles of $S^{2n}$ (see Browder [1, p. 122]). The boundary of $M_0^{4n}$ is the PL sphere $S^{4n-1}$. We set $M^{4n} = M_0^{4n} \cup_{\partial} CS^{4n-1}$ to obtain a closed PL manifold of index 8.

In the rest of this note, $P(X)$, $E(X)$ and $\Gamma(X)$ will denote the polynomial algebra, exterior algebra, and divided power algebra, respectively generated by the set $X$. For a natural number $n$, $\alpha(n)$ will be the number of nonzero terms in the dyadic expansion and $r(n)$ the 2-adic valuation ($n = 2^{r(n)}$ odd).


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THEOREM A. As rings,

\[(\Omega^\text{TOP}_*/\Tor) \otimes Z_{(2)} = P\{[CP^{2n}] \mid \alpha(n) < \psi(n) + 4\} \otimes \Gamma\{[M^{4n}] \mid \alpha(n) \geq \psi(n) + 4\}.\]

Moreover, \((\Omega^\text{PL}_*/\Tor) \otimes Z_{(2)} = (\Omega^\text{TOP}_*/\Tor) \otimes Z_{(2)}.\) Here \(CP^{2n}\) is the complex projective space.

The torsion structures of \(\Omega^\text{TOP}_* \otimes Z_{(2)}, \ast \neq 4\) and \(\Omega^\text{PL}_* \otimes Z_{(2)}\) are very involved, and even though our techniques give the groups, we know comparatively little about the explicit generators. However, there are a finite number of explicit constructions—twisted products, and Massey products—which generate the torsion from a small set of "basic" torsion manifolds. Among these generators are specific ones given by relations among the Milnor manifolds and the \(CP^{2n}\)'s. For example, the relation below (the first which occurs) generates a \(\Z/2\Z\) direct summand in \(\Omega^\text{PL}_8\).

1.2 \[2\{7[M^8] - 200[CP^8 \times CP^8] + 144[CP^8]\} = 0\]

while in dimension 12 there is a \(\Z/4\Z\) summand generated by the relation

1.3 \[4\{31[M^{12}] - 1620[CP^{12}] + 5292[CP^8] \cdot [CP^8] - 3920[CP^8]^3\} = 0.\]

1.2 and 1.3 are a little surprising since it is well known that the smallest multiple of \(M^8\) which is actually PL homeomorphic to a differentiable manifold is \(28M^8\) while the corresponding number for \(M^{12}\) is 992.

In the rest of this note, all spaces and maps are to be taken in the 2-local category (see [10] for a precise definition). Unless otherwise indicated \(H_\ast(X) (H^\ast(X; Z))\) will denote homology (cohomology) of \(X\) with \(Z\) coefficients. \((\text{Note. } H_\ast(X; Z) = H_\ast(X; Z_{(2)})\) when \(X\) is 2-local.\)

2. Preliminaries. The map \(B(G/TOP) \rightarrow B(G/TOP).\) It is a well-known result of Sullivan that \(G/TOP\) is a product of Eilenberg-Mac Lane spaces. In [7] and [8] specific homotopy equivalences

\[K: G/TOP \rightarrow \prod_{n \geq 1} K(Z_{(2)}, 4n) \times K(Z/2, 4n - 2)\]

were constructed. The mapping \(K\) depends on the "genus" used in the "surgery formulas". In this note we use the map defined in [7].

In [6] we examined the space \(B(G/TOP)\) as well as the natural map \(B\pi_\ast : BSG \rightarrow B(G/TOP).\) The main result there is

PROPOSITION 2.1. (i) There is an H-map

\[BK: B(G/TOP) \rightarrow \prod_{n \geq 1} K(Z_{(2)}, 4n + 1) \times K(Z/2, 4n - 1)\]
with $\Omega(BK \circ B\pi) = K \circ \pi$ and $BK$ a homotopy equivalence ($\pi: SG \to G/\text{TOP}$ the natural map).

(ii) The class $B\pi^*(K_{4n+1})$ is divisible by precisely $2^{a(n)-1}$, where $K_{4n+1} = (BK)^*$ (fundamental class).

Next we specify the classes $(B\pi)^* K_{4n+1}$ more precisely. To do this we will specify the structure of the $Z_{(2)}$ cohomology of $BSG$ by determining its Bochstein spectral sequence (BSS). We first introduce 3 (acyclic) $DG$-Hopf algebras over $Z_{(2)}$ which will be our basic building blocks.

(I) \[ A_0(k) = P\{p_n \mid n \geq 1\} \otimes E\{e_n \mid n \geq 1\}, \]
\[ \deg(p_n) = 4n, \quad \deg(e_n) = 4n + 1, \quad \psi(p_n) = \sum p_i \otimes p_{n-i}, \]
\[ \psi(e_n) = \sum p_i \otimes e_{n-i} + e_i \otimes p_{n-i}, \quad \delta(p_n) = 2^k e_n. \]

(II) \[ A_1\{x \mid k\} = P\{x\} \otimes E\{y\}, \]
\[ \deg x = 4n, \quad \deg y = 4n + 1, \quad \psi(x) = 1 \otimes x + x \otimes 1, \]
\[ \psi(y) = 1 \otimes y + y \otimes 1, \quad \delta x = 2^k y. \]

(III) \[ A_2\{x \mid k\} = E\{y\} \otimes \Gamma(x), \]
\[ \deg x = 4n, \quad \deg y = 4n - 1, \quad \psi(y) = 1 \otimes y + y \otimes 1 \]
and
\[ \psi(x) = 1 \otimes x + x \otimes 1, \quad \delta y = 2^k x \]
(hence $\delta(y \cdot \gamma_{2r-1}(x)) = 2^{k+r} \gamma_{2r}(x)$). If $X$ is a graded set concentrated in degrees congruent to zero mod 4, we write $A_i\{X \mid k\} = \otimes_{x \in X} A_i\{x \mid k\}$, $i = 1, 2$. Each of the $DG$-Hopf algebras above have an associated Bochstein spectral sequence $\{E_r(\ ), d_r\}$. From [5] we quote

**Proposition 2.2.** For $r \geq 2$, the cohomology BSS of the space $BSG$ is

\[ E_r(BSG) = E_r(A_0(3)) \otimes E_r(A_2\{X \mid 2\}) \]

for a suitable graded set $X$.

Let $j_r: H^*(BSG) \to E_r(BSG)$ denote the natural reduction map. From [3] and [6] we have

**Proposition 2.3.** (i) $j_{3}(2^{1-a(n)}B\pi^*(K_{4n+1})) = e_n + \text{decomposable terms.}$

(ii) $B\pi^*(K_{4n-1}) = 0$ for $a(n) > 1$.

(iii) $Sq^2 B\pi^*(K_{2l-1}) = e_{2l-1}$. 

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3. The DG-Hopf algebra $\mathcal{F}$. In §4 we show that the following DG-
Hopf algebra over $\mathbb{Z}_{(2)}$ is a split subalgebra of the BSS for BSTOP.

$$\mathcal{F} = P\{p_n \mid n \geq 1\} \otimes P\{k_n \mid n \geq 1\} \otimes E\{\varepsilon_n \mid n \geq 1\},$$

$$\deg p_n = 4n, \quad \deg k_n = 4n \quad \text{and} \quad \deg \varepsilon_n = 4n + 1,$$

$$\psi(p_n) = \sum p_i \otimes p_{n-i},$$

$$\psi(k_n) = 1 \otimes k_n + k_n \otimes 1, \quad \psi(\varepsilon_n) = 1 \otimes \varepsilon_n + \varepsilon_n \otimes 1,$$

with differential structure given by

$$\delta p_n = 16\varepsilon_n, \quad \delta k_n = 2^{2(n)}\varepsilon_n \quad \text{where} \quad \varepsilon_n = \sum \varepsilon_i p_{n-i}.$$ 

Husemoller [4] has introduced a splitting of the Hopf algebra $P\{p_n \mid n \geq 1\}$ as a tensor product of “smaller” Hopf algebras,

$$P\{p_n \mid n \geq 1\} = \otimes_{n \text{ odd}} P\{p_{n,0}, p_{n,1}, \cdots, p_{n,i}, \cdots\}$$

$$(\deg p_{n,i} = 2^{i+2}n).$$

We split $\mathcal{F}$ accordingly,

$$\mathcal{F} = \otimes_{n \text{ odd}} \mathcal{F}(n),$$

$$\mathcal{F}(n) = P\{p_{n,0}, p_{n,1}, \cdots\} \otimes P\{k_{n,0}, k_{n,1}, \cdots\} \otimes E\{\varepsilon_{n,0}, \varepsilon_{n,1}, \cdots\}.$$ 

Here $k_{n,i} = 2^{2n}e_{n,i}$, $\varepsilon_{n,i} = \varepsilon_{n,2i}$ and the differential structure is (inductively) determined by

$$\delta(k_{n,i}) = 2^{2(n)}\varepsilon_{n,i} \quad \text{and} \quad \delta(2^i p_{n,i} + \cdots + p_{n,0}^a) = 2^{i+4}e_{n,i}.$$ 

**Lemma 3.1.** (i) If $\alpha(n) < 4$, then

$$E_s(\mathcal{F}(n)) = P\{p_{n,0}, p_{n,1}, \cdots\} \otimes E_s(A_1\{k_{n,0}, k_{n,1}, \cdots \mid \alpha(n)\}).$$

(ii) If $\alpha(n) \geq 4$, then for $s \geq \alpha(n)$,

$$E_s(\mathcal{F}(n)) = P\{k_{n,0}, \cdots, k_{n,0}, k_{n,r}, k_{n,r+1}, \cdots\}$$

$$\otimes E_s(A_1\{k_{n,r}, k_{n,r+1}, \cdots \mid \alpha(n)\}),$$

where

$$r = \alpha(n) - 4 \quad \text{and} \quad k_{n,r+i} = p_{n,i}^a + \sum_{j=1}^{r-1} p_{n,i-j-1}^a p_{n,i-j}^a k_{n,r+i-j} + k_{n,r+i}.$$ 

4. **Theorem A.** There is a natural map $\text{BSO} \times G/TOP \to \text{BSTOP}$
which on homology leads to

$$P\{a_n \mid n \geq 1\} \otimes \Gamma\{b_n \mid n \geq 1\} \xrightarrow{r^*} H_*(\text{BSTOP})/\text{Tor},$$

where $a_n$ is dual to $p^n_1 \in H^n(\text{BSO})/\text{Tor}$ and $b_n$ is spherical. We observe that the structure of $H_*(\text{BSTOP})/\text{Tor}$ follows at once if we can prove that

$$(H^*(\text{BSTOP})/\text{Tor}) \otimes \mathbb{Z}/2 = E_\alpha(\mathcal{F}), \quad \text{where} \quad E_\alpha(\mathcal{F}) = \otimes_{n \text{ odd}} E_\alpha(\mathcal{F}(n))$$

is
described in 3.1. Therefore the thrust of the argument is to evaluate the BSS of BSTOP.

Our starting point is the fibration sequence, \( \cdots \rightarrow \text{BSTOP} \rightarrow \text{BSG} \rightarrow \text{B}(G/\text{TOP}) \rightarrow \cdots \). It is convenient to decompose this sequence in two steps. Let

\[
\text{BK}_1 = \prod_{i>1} K(\mathbb{Z}/2, 2^i - 1)
\]

and

\[
\text{BK}_2 = \prod_{n>1} K(\mathbb{Z}_{(2)}, 4n + 1) \times \prod_{a(n)>1} K(\mathbb{Z}/2, 4n - 1).
\]

We have the fibration sequences (\( \Omega \text{BK}_i = \text{K}_i \))

\[
\cdots \rightarrow K_1 \rightarrow BX \rightarrow \text{BSG} \rightarrow K_1 \rightarrow \cdots
\]

\[
\cdots \rightarrow K_2 \rightarrow \text{BSTOP} \rightarrow BX \rightarrow K_2 \rightarrow \cdots.
\]

**Lemma 4.3.** (i) There are graded sets \( X_1 \) and \( X_2 \) such that for \( r \geq 2 \) the \( r \)th term in the BSS of \( BX \) is

\[
E_r(BX) = E_r(A_0(4)) \otimes E_r(A_1(X_1 | 2)) \otimes E_r(A_2(X_2 | 2)).
\]

(ii) The inclusion \( i: K_1 \rightarrow BX \) maps \( E_r(A_1(X_1 | 2)) \) injectively into BSS for \( K_1 \).

It follows from 2.5 and 4.3 above that

\[
H^*(\text{BSTOP}; \mathbb{Z}/2) = H^*(BX; \mathbb{Z}/2) \otimes H^*(K_2).
\]

Let \( j: K_2 \rightarrow \text{BSTOP} \) be the map in 4.2. Our main technical result is

**Theorem 4.4.** (i) There are graded sets \( Y_1 \) and \( Y_2 \) such that for \( r \geq 2 \)

\[
E_r(\text{BSTOP}) = E_r(\mathcal{F}) \otimes E_r(A_1(Y_1 | 2)) \otimes E_r(A_2(Y_2 | 2)).
\]

(ii) \( j^* \) maps \( E_r(A_1(Y_1 | 2)) \) monomorphically to the BSS for \( \prod K(\mathbb{Z}_{(2)}; 4n) \times \prod_{a(n)>1} K(\mathbb{Z}/2; 4n-2) \).

We first give an exact sequence of spectral sequences,

\[
\mathbb{Z}/2 \rightarrow E_r(A_1(Y_1 | 2)) \otimes E_r(A_2(Y_2 | 2)) \rightarrow E_r(\text{BSTOP}) \rightarrow \hat{E}_r \rightarrow \mathbb{Z}/2,
\]

satisfying (ii) and with \( \hat{E}_r = E_r(\mathcal{F}) \). From dimensional considerations and because \( j^*(k_n) \) is an infinite cycle and \( j^*(p_n)=0 \), it follows that this sequence splits:

\[
E_r(\text{BSTOP}) = \hat{E}_r \otimes E_r(A_1(Y_1 | 2)) \otimes E_r(A_2(Y_2 | 2)).
\]

Algebraic considerations lead to the pleasant fact that \( \hat{E}_\infty \) is a polynomial algebra with one generator in each degree congruent to zero mod 4.
Since

\[ E_\infty = E_\infty(B\text{STOP}) = H^*(B\text{STOP}) / \text{Tor} \otimes \mathbb{Z}/2 \]

we see that \( H^*(B\text{STOP}) / \text{Tor} \) is a polynomial algebra. In particular the 4\( \ast \)-dimensional primitives of \( H_\ast(B\text{STOP}) / \text{Tor} \) are a copy of \( \mathbb{Z}_2 \).

We now employ a result of Morgan and Sullivan [8]. They construct a class \( L_n \in H^{4n}(B\text{STOP}) \) whose rational reduction is the (inverse) Hirzebruch class when restricted to \( H^{4n}(\text{BSO}; \mathbb{Q}) \) and whose restriction to \( G/\text{TOP} \) is 8 ("surgery class"). Since the coefficient of \( p_n \) in the Hirzebruch class is \( 2^{a(n)-1} \) (odd), it follows that

\[ 2^{\pi(n)-1} \cdot \tau_{a}(b_n) = 8 \cdot \tau_{a}(s(a_1, \ldots, a_n)). \]

(\( s_n \) is the Newton polynomial.)

This equation implies that \( \tau_{a}(y_2(b_n)) \) is divisible by 2 unless \( a(n) \geq 4 + v(n) \), and from this one can inductively conclude that

\[ \hat{E}_\ast = E_\ast(T). \]

REFERENCES


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