Let $S$ be a commutative ring with identity. Let $T$ be a commutative $S$-algebra. $A_S(T)$ denotes the higher differential algebra over $S$, in the sense of Berger [1] or Kawahara-Yokoyama [5], with an index set consisting of all nonnegative integers. \( \{d^n_{T/S}\}_{n=0,1,2,\ldots} \) denotes the canonical higher $S$-derivation of $T$ into $A_S(T)$. In case $S$ is the ring of all integers, we use simplified notations $A(T)$ and $d^n_T$ ($n=0,1,2,\ldots$).

Let $R$ denote a complete discrete valuation ring, of a valuation $v$ of unequal characteristic with maximal ideal $m=(\pi)$. Assume that the characteristic of $k=R/m$ is $p$. Let $P$ be a coefficient ring of $R$. Let $\{\tilde{e}_i\}_{i\in\Gamma}$ be a $p$-independent base of $k$ and let $c_i$ be a representative of $\tilde{e}_i$, chosen from $P$ for every $i\in\Gamma$. The symbol $^\wedge$ means the $p$-adic completion of $P$-algebra. By arguments developed by Berger or Kawahara-Yokoyama in the cited papers, and formal smoothness and flatness of $P$ over the prime local ring, we can deduce the following theorem.

**Theorem 1.** $A(P)^\wedge = P[d^nc_{i}]_{i\in\Gamma; n=0,1,2,\ldots}$ is a polynomial ring over $P$ in distinct indeterminates $d^nc_i$'s.

For simplicity, we denote canonical images of $d^nc_i$ in $A(R)^\wedge$ by the same notation $d^nc_i$, for $i\in\Gamma$. Let $\{d^n\}_{n=0,1,2,\ldots}$ be the higher derivation of the polynomial ring $P[X]$ into $(R \otimes_{P[X]} A(P[X]))^\wedge$. Let $f(X)$ be the Eisenstein polynomial over $P$ such that $f(\pi)=0$. Then we have the following formula for every $n\geq 1$.

\[
d^nf(X) = f'(\pi) d^nX + \sum_{j=2}^{\infty} f^{(j)}(\pi) \sum_{j_1+\cdots+j_l=j} d^{i_1}X \cdots d^{i_l}X
\]

\[
+ pG_n(d^1X, d^2X, \cdots; \cdots, d^lc_i, \cdots),
\]

where the second sum of the middle term is taken for the sets of integers.


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of indeterminates \( \{Z_k, W_{i,i}\} \); \( i = 1, 2, \ldots \) of weight \( n \), when we define as weight \( Z_i = weight W_{i,i} = i \), such that for a given integer \( i > 0 \) all but a finite number of coefficients of \( G_n \) belong to \( \mathfrak{m}^i \) and \( G_n \) has no terms consisting of monomials of only \( Z_k \)’s.

Solving equations \( d_1 f(X) = 0 \), \( d_2 f(X) = 0 \), \( \ldots \) successively, we obtain relations

\[
(f'(\pi))^{2n-1} d^n_R \pi = F_n(\pi, \ldots, d^n_R \pi, \ldots),
\]

where \( F_n(\pi, \ldots, W_{i,i}, \ldots) \) have properties similar to \( G_n \).

As an extended notion of Neggers’ number for derivations of order 1 in Neggers [6] and Suzuki [7], [8], we give

**Definition.** \( \Delta_P^\pi(\pi) = \min \{ \text{coefficient of } F_n(\pi) - (2n-1)n(f'(\pi)) \} \) is called the \( n \)th Neggers number for \( (P, \pi) \), \( n = 1, 2, \ldots \).

We can show that \( \Delta_P^\pi(\pi) \) is independent of the choice of \( \{c_i\}_{i \in \mathbb{I}} \).

Henceforth for a higher derivation \( \{\partial^n\}_{n=0,1,2,\ldots} \) of \( P \) into \( P \), \( R \) into \( R \) or \( k \) into \( k \), we always assume that \( \partial^0 = \text{the identity map} \).

**Theorem 2.** The following four conditions are equivalent.

(i) Every higher derivation of \( P \) into \( P \) is extended to a higher derivation of \( R \) into \( R \).

(ii) \( \Delta_P^\pi(\pi) \geq 0 \) for all \( n = 1, 2, \ldots \).

(iii) \( \Delta_P^\pi(\pi) \leq 1 \) for all \( n = 1, 2, \ldots \).

(iv) Every higher derivation of \( k \) into \( k \) is induced by a higher derivation of \( R \) into \( R \).

**Outline of the proof.** The crucial point of the proof of this theorem is to show that (ii) implies (iii). Assume that (ii) is true and (iii) is false. Let \( n \) be the least integer such that \( \Delta_P^\pi(\pi) = 0 \). Let \( e \) be the degree of \( f(X) \).

We can show that there exists a higher derivation \( \{\partial^n\}_{n=0,1,2,\ldots} \) of \( R \) into \( R \) such that \( \partial^n \pi \in \mathfrak{m} \). On the other hand, by the expansion of \( (\partial^nf)(\pi) \) according to (1) we can show that \( \partial^n \pi \in \mathfrak{m} \), which is a contradiction.

If \( k \) is perfect or if \( R \) is tamely ramified, \( R \) satisfies conditions in Theorem 2.

**Theorem 3.** If \( R \) satisfies conditions in Theorem 2, the ideals \( (f'(\pi)), (f''(\pi)/2!), \ldots, (f^{(n)}(\pi)/n!) \) are independent of the choice of \( P \) and \( \pi \) for all \( n = 1, 2, \ldots \).

**Outline of the proof.** Let \( \psi \) be an isomorphism of \( P \) onto another coefficient ring \( P' \) which induces an identity map on \( k \). Expressing \( \psi \)
as a sum of all components of a higher derivation as in Heerema [3], it can be shown that $\psi$ is extended to an automorphism $\lambda$ of $R$ by (i). $\psi$ is extended to an isomorphism of $A_P(R)$ onto $A_P(R)$. $A_P(R)$ and $A_P(R)$ being graded algebras, we compare Fitting ideals of $R$-submodules of grade $n$ of both algebras and deduce our theorem.

REFERENCES


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