FACTORIZATION AND INVARIANT SUBSPACES
FOR NONCONTRACTIONS

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1. Introduction. The purpose of this note is to announce a generalization of the Sz.-Nagy-Foiaş model theory for contractions to arbitrary bounded operators. We also indicate how invariant subspaces are described by this model theory.

The Russians, for example, Livšic [14] and Brodskiï and Livšic [7], have studied model theories for various classes of operators, often including some noncontractions. Recently there has been some work, for example, Davis and Foiaş [13], Brodskiï, Gohberg, and Kreĭn [9], and Brodskiï [8] on characteristic functions for noncontractions. Our work is closely related to that of Clark [11] and depends heavily for inspiration upon the canonical models of de Branges-Rovnyak [5].

In many of these papers, one of the main points is the connection between factorizations of the characteristic function $B$ and invariant subspaces. Sz.-Nagy-Foiaş [15] found a precise condition on a factorization of $B$ to insure that it results (for contractions) in an invariant subspace. Also the work of de Branges [4], [5] should be mentioned. Most recently Clark [12] has taken this problem up for invertible noncontractions. We propose to study this problem for the class of bounded noncontractions.

2. Model theory. The characteristic operator function $B(z)$ of a bounded Hilbert space operator $T$ is defined by

$$B(z) = -TJ_T + z |I - TT^*|^{1/2} (I - zT^*)^{-1} |I - T^*T|^{1/2}$$

where $J_T = \text{sgn}(I-T^*T)$ and where $B$ acts from $\mathcal{D}_T$, the closure of the range of $|I-T^*T|^{1/2}$, to $\mathcal{D}_{T*}$. A basic problem of model theory is to construct from $B$, in a canonical way, a bounded operator $T$ such that $B$ satisfies (1).

Let $\mathcal{C}_*$ and $\mathcal{C}$ be Hilbert spaces, and let $B(z): \mathcal{C} \rightarrow \mathcal{C}_*$ be analytic in a neighborhood $D$ of 0. We also assume that $D$ is symmetric about the real line. Let

$$J = \text{sgn}(I - B(0)*B(0)),$$
$$J_* = \text{sgn}(I - B(0)B(0)^*),$$
$$\text{sgn } 0 = 1.$$
Let \( \hat{B}(z) = B(\bar{z})^* \). For each \( c \in \mathcal{C}_* \), \( d \in \mathcal{C} \), let
\[
k(w, z)(c, d)
\]
(2) \[= ([J* - B(z)J B(w)^*]c/(1 - z\bar{w}) + [B(z) - B(\bar{w})]d/(z - \bar{w}),
\]
\[
[B(z) - B(\bar{w})]Jc/(z - \bar{w}) + [J - B(z)J* B(w)^*]c/(1 - z\bar{w}));
\]
then \( k(w, z)(c, d) \) is an ordered pair of functions each analytic in \( z \) on \( D \), the first component taking values in \( \mathcal{C}_* \), the second in \( \mathcal{C} \). Let \( H_0 = \{ k(w, z)(c, d) | w \in D, c \in \mathcal{C}_*, d \in \mathcal{C} \} \) and \( H_1 \) be all linear combinations of elements of \( H_0 \). For \( k(w_1, z)(c_1, d_1) \) and \( k(w_2, z)(c_2, d_2) \) two elements of \( H_0 \), define

(3) \[ \langle k(w_1, z)(c_1, d_2), k(w_2, z)(c_2, d_2) \rangle = \langle k(w_1, w_2)(c_1, d_1), (c_2, d_2) \rangle,
\]
the second inner product taken in \( \mathcal{C}_* \times \mathcal{C} \), the elements of which are written as row vectors. Extend (3) to \( H_1 \) by linearity. It can be shown (see [1]) that a necessary condition that \( B(z) \) be a characteristic function is that (3) be positive-definite.

When this is the case, we let \( \mathcal{D}(B) \) (following notation of de Branges-Rovnyak) be the completion of the pre-Hilbert space \( H_1 \). The elements of \( \mathcal{D}(B) \) can be taken to be of the form \((f(z), g(z))\), where both \( f \) and \( g \) are analytic on \( D \), \( f \) is valued in \( \mathcal{C}_* \), \( g \) in \( \mathcal{C} \). When convenient, we write \((f, g)\) rather than \((f(z), g(z))\).

Define a linear operator \( S: H_0 \rightarrow H_1 \) by
\[
S: k(w, z)(c, d) \rightarrow \tilde{w}^{-1}[k(w, z)(c, 0) - k(0, z)(c_0, 0)] + \tilde{w}k(w, z)(0, d) - k(0, z)(J* B(\tilde{w})d, 0)
\]
and extend \( S \) to \( H_1 \) by linearity. The following theorem (proved in [1]) shows that this construction yields a model for a general bounded operator.

**Theorem 1.** The operator \( S \) extends by continuity to a bounded operator (also \( S \)) on \( \mathcal{D}(B) \), and \( B \) coincides with the characteristic function of \( S \). A formula for \( S \) independent of a kernel function representation is
\[
S: (f(z), g(z)) \rightarrow (zf(z) - B(z)J g(0), [g(z) - g(0)]/z) \quad \text{and} \quad S^*: (f(z), g(z)) \rightarrow ([f(z) - f(0)]/z, zg(z) - B(z)J* f(0)).
\]

Also proved in [1] are the useful relations
\[
I - SS^* = e_1(0)^*J* e_1(0), \quad I - S*S = e_2(0)^*J e_2(0),
\]
where \( e_1(0): \mathcal{D}(B) \rightarrow \mathcal{C}_* \) is defined by \((f, g) \rightarrow f(0)\) and \( e_2(0): \mathcal{D}(B) \rightarrow \mathcal{C} \) is given by \((f, g) \rightarrow g(0)\). It follows from the definition of \( \mathcal{D}(B) \) that
\[
e_1(0)^*c = k(0, z)(c, 0) \quad \text{and} \quad e_2(0)^*d = k(0, z)(0, d).
\]
A consequence of Theorem 1 is that the positive-definiteness of the bilinear form (3) is necessary and sufficient for \( B \) to be a characteristic operator function. There results a new proof of the theorem of Brodskiï [8]. Clark [11] and Brodskiï, Gohberg and Kreïn [9] handle the case where \( B(z) \) is invertible in \( D \).

3. Invariant subspaces. We restrict ourselves, for the purpose of studying invariant subspaces, to factorizations we call standard.

DEFINITION 1. For \( B(z) : \mathbb{C}_1 \rightarrow \mathbb{C}_3 \) a characteristic operator function, the factorization \( B = B_2 \cdot B_1 \) \((B_1(z) : \mathbb{C}_1 \rightarrow \mathbb{C}_2, B_2(z) : \mathbb{C}_2 \rightarrow \mathbb{C}_3)\) is said to be standard if:

(i) \( B_2 \) and \( B_1 \) are also characteristic operator functions.
(ii) On \( \mathbb{C}_1, J_1 \equiv \text{sgn } I - B_1(0)^*B_1(0) = \text{sgn } I - B(0)^*B(0) \).

On \( \mathbb{C}_2, J_2 \equiv \text{sgn } I - B_2(0)^*B_2(0) = \text{sgn } I - B_1(0)B_1(0)^* \).

On \( \mathbb{C}_3, J_3 \equiv \text{sgn } I - B_2(0)B_2(0)^* = \text{sgn } I - B(0)^*B(0)^* \).

Let \([c, d], = (J, c, d)\) be the associated indefinite inner product on \( \mathbb{C}_i \).

Call an operator \( X : \mathbb{C}_i \rightarrow \mathbb{C}_k \) J-contractive if \([Xc, Xc]_k \leq [c, c]_i, i, k = 1, 2, 3\). Then the condition for a factorization to be standard is essentially that \( B(z) \) be the product of J-contractions. This generalizes the situation in the contraction case, where representations of \( B \) as a product of contractions is studied (Sz.-Nagy-Foiaş [15]).

If \( B(z) \) is of the form (1), we show that standard factorizations correspond to invariant subspaces, if not for \( T \), then for \( T \oplus U \), where \( U \) is a unitary operator.

THEOREM 2. Let \( B = B_2 \cdot B_1 \) be a standard factorization. Then there is a partial isometry \( \Gamma \) from \( D(B_2) \oplus D(B_1) \) onto \( D(B) \) given by

\[
\Gamma : (f_2, g_2) \oplus (f_1, g_1) \rightarrow (f_2 + B_2f_1, B_1g_2 + g_1).
\]

The difficulty for invariant subspaces is that \( \Gamma \) may have a nontrivial kernel. The situation is best understood by defining another space, the overlapping space of de Branges and Rovnyak.

DEFINITION 2. Let \( B = B_2 \cdot B_1 \) be a standard factorization. Define a space \( \mathcal{E} = \mathcal{E}(B_2 \cdot B_1) \) by

\[
\mathcal{E} = \{(f, g) \mid (B_2f, -J_2g) \in D(B_2) \text{ and } (f, -B_1J_2g) \in D(B_1)\}
\]

with a norm given by

\[
\|(f, g)\|_{\mathcal{E}}^2 = \|(B_2f, -J_2g)\|_{D(B_2)}^2 + \|(f_1, -B_1J_1g)\|_{D(B_1)}^2.
\]

THEOREM 3. (i) \( \mathcal{E} \) is isometrically isomorphic to \( \mathcal{N} \) = the kernel of \( \Gamma \) (see Theorem 2) under the map \( \chi : (f, g) \rightarrow (B_2f, -J_2g) \oplus (-f, B_1J_2g) \);

(ii) the operator \( U \) defined by

\[
U : (f(z), g(z)) \rightarrow (zf(z) + g(0), [g(z) - g(0)]/z)
\]
is unitary on $\mathcal{E}$ with adjoint

$$U^* : (f(z), g(z)) \mapsto ([f(z) - f(0)]/z, zg(z) + f(0)).$$

Note that (i) follows directly from the definitions. The proof of (ii) is a direct computation, using relations (4) and (5) in the appropriate spaces.

It follows from (ii) and Theorem 1 of de Branges [2] that $\mathcal{E}$ is a space of the type $\mathcal{E}(q)$ studied by de Branges and Rovnyak [5].

The above analysis gives rise to an invariant subspace theorem, known to the Russians in terms of a somewhat different model theory [6].

**Theorem 4.** Let $B = B_2 \cdot B_1$ be a standard factorization. Let $S$, $S_1$ and $S_2$ be the model operators in $\mathcal{D}(B)$, $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$ respectively, and let $U$ be the unitary operator of Theorem 3 in $\mathcal{E}(B_2 \cdot B_1)$. Then

$$\Gamma' = (f_2, g_2) \oplus (f_1, g_1)$$

is unitary from $\mathcal{D}(B_1) \oplus \mathcal{D}(B)$ onto $\mathcal{D}(B) \oplus \mathcal{E}$, where $\mathcal{N}$ and $\chi$ are as in Theorem 3, (ii) $\mathcal{M} = \Gamma'((0) \oplus \mathcal{D}(B_1))$ is an invariant subspace for $S \oplus U$; $S \oplus U \mathcal{M}$ is unitarily equivalent to $S_1$ via $\Gamma'$, (iii) $\mathcal{M}^\perp = \Gamma'(\mathcal{D}(B_2) \oplus (0))$ is invariant for $S^* \oplus U^*$; $S^* \oplus U^*) \mathcal{M}^\perp$ is unitarily equivalent to $S_2^*$ via $\Gamma'$.

We hope to publish details elsewhere.

**References**


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