ON IDEALS OF COMPACT OPERATORS

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In [1] Brown, Pearcy and Salinas give an affirmative answer to the following question: Given a compact operator $T$ on a separable Hilbert space $H$, is there an ideal $\Lambda(T)$ containing $T$ and different from the ideal $K$ of all compact operators? Their construction relies on some ideas of Von Neumann-Calkin [2] and is rather complicated.

The purpose of this brief note is to show that the existence of such a $\Lambda(T)$ follows from elementary properties of the $s$-numbers of $T$. Recall that the $s$-numbers, $(s_n(T))$, are the eigenvalues of $(TT^*)^{1/2}$ arranged in decreasing order and counting multiplicities. We list the three properties we need:

1. If $S, T \in K$, then $s_{n+m}(S+T) \leq s_n(S) + s_m(T)$.
2. If $R, S, T \in L$, then $s_n(RST) \leq \|R\|s_n(S)\|T\|$. Here $L$ denotes the space of all bounded linear operators on $H$.
3. If $T \in K$, then there are orthonormal sets $(f_n)$ and $(y_n)$ in $H$ such that $r = \sum_{n=1}^{\infty} s_n(T)f_n \otimes y_n$.

We also use the following fact concerning real sequences:

4. If $(\beta_n)$ is a nonnegative sequence of real numbers increasing to $\infty$ with $1/\beta_n \in c_0 \setminus \bigcup_{p>0} l_p$, then there is a positive sequence $(\alpha_n) \in l_1$ such that $\sum_{n=1}^{\infty} \alpha_n \beta_n = +\infty$ and $(\beta_n \alpha_n)$ is decreasing. Also, there is a decreasing null sequence $(\gamma_n)$ such that $\sum_{n=1}^{\infty} \gamma_n \alpha_n \beta_n = +\infty$.

The construction of $\Lambda(T)$. Let $\sigma_p = \{T \in K: \sum_{n=1}^{\infty} s_n(T)^p < +\infty\}$. It is well known and easy to prove that $K \setminus \bigcup_{p>0} \sigma_p \neq \emptyset$. (For a study of these important ideals see [3] and [4].) Thus we may suppose, for our purpose, that $T \in K \setminus \bigcup_{p>0} \sigma_p$. Then $\beta_n = 1/s_n(T)$ increases to $\infty$ and $1/\beta_n \in c_0 \setminus \bigcup_{p>0} l_p$. Let $(\alpha_n)$ be as in (4) and let

$$\Lambda(T) = \left\{ S \in L: \sum_{n=0}^{\infty} s_n(S) \alpha_n \beta_n < +\infty \right\}.$$ 

It follows from (4) and the definition of $(\beta_n)$ that $T \in \Lambda(T)$.


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If the rank of $S$ is finite, then $s_n(S) = 0$ for large enough $n$ and so $\Lambda(T)$ contains the finite rank operators (this fact also follows from deeper considerations).

If $R, S \in \Lambda(T)$ then by (1) and (4)

$$\alpha_{2n}^n \beta_{2n} s_{2n}(R + S) \leq \alpha_n \beta_n s_n(R) + \alpha_n \beta_n s_n(S),$$

and similarly for $\alpha_{2n-1}^n \beta_{2n-1} s_{2n-1}(R + S)$. Thus $\Lambda(T)$ is a linear space. Also, it follows from (2) that $\Lambda(T)$ is closed under left and right composition by bounded linear operators.

Let $T_0 = \sum \gamma_n f_n \otimes y_n$, where $(\gamma_n)$ is as in (4) and $(f_n)$ and $(y_n)$ are determined by (3). Then $s_n(T_0) = \gamma_{n+1}$ and so by (4) $T_0 \in K \setminus \Lambda(T)$.

Using the recent results [5] and [6] it is easy to generalize the above construction to include many other classes of Banach spaces.

Recall that the $n$th approximation numbers, $\alpha_n(T)$, of $T$ are given by

$$\alpha_n(T) = \inf \{ \|T - A\| : \text{rank } A \leq n \}.$$

For Banach spaces $E, F$ we say $T \in l_p(E, F)$ if $\sum_{n=1}^{\infty} \alpha_n(T)^p < +\infty$. These generalized ideals have been studied by Pietsch and others. The only fact we need here is the following [5]: If $K(E, F) = l_p(E, F)$ for some $p > 0$, then $\min(\dim E, \dim F) < +\infty$. Here $K(E, F)$ denotes the compact operators from $E$ to $F$. We also recall the following definition [6]: Two Banach spaces $E$ and $F$ form a Bernstein pair if for any positive, decreasing null sequence $(b_n)$ there is a $T \in K(E, F)$ such that

$$0 < \inf_n \frac{\alpha_n(T)}{b_n} \leq \sup_n \frac{\alpha_n(T)}{b_n} < +\infty. \quad (5)$$

Let $\mathcal{K}$ denote the ideal of all compact operators between arbitrary Banach spaces and let $T \in \mathcal{K}$.

**Theorem.** There exists a complete quasi-normed ideal $\Lambda(T)$ such that $\Lambda(T)(E, F) \neq K(E, F)$ whenever $(E, F)$ forms a Bernstein pair.

Indeed, by the result of Pietsch, we may assume that $(\alpha_n(T)) \in c_0 \setminus \bigcup_{p > 0} l_p$. Let $\beta_n = 1/\alpha_n(T)$ and let $\alpha_n, \gamma_n$ be as in (4) above. Let

$$\Lambda(T) = \left\{ S \in \mathcal{K} : \sum_{n=1}^{\infty} \alpha_n(S) \alpha_n \beta_n < +\infty \right\}.$$

Since the approximation numbers have properties (1)–(3), $\Lambda(T)$ is ideal. It is easy to show that under the quasi-norm

$$\rho(S) = \sum_{n=1}^{\infty} \alpha_n(S) \alpha_n \beta_n,$$
$\Lambda(T)$ is complete. If $\langle E, F \rangle$ is a Bernstein pair, then by (5) there is an $S \in K(E, F)$ such that $\inf_n (\alpha_n(S)/\gamma_n) > 0$ and thus, by (4), $S \notin \Lambda(T)(E, F)$.

We remark that all classical Banach spaces form Bernstein pairs. In particular $\langle L_p(\mu), L_q(\gamma) \rangle$ is a Bernstein pair for all $1 \leq p, q \leq \infty$ and measures $\mu, \gamma$ [6].

**BIBLIOGRAPHY**


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