SOME THEOREMS ON C-FUNCTIONS

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The purpose of this note is to announce certain results I have obtained about the behavior of the Harish-Chandra C-function as a meromorphic function. The notation and terminology, if not explained, are that of [2], [3], or [6].

1. The C-ring. Let \((P, A)\) be a fixed parabolic pair of a semisimple Lie group \(G\) having finite center, \(P=MAN\) the corresponding Langlands decomposition, \(K\) a fixed maximal compact subgroup. Let \(\mathfrak{g}, \mathfrak{K}, \mathfrak{K}_M, \mathfrak{M}\) be the universal enveloping algebras of \(G, K, K_M,\) and \(M,\) respectively \((K_M=K\cap M)\) — i.e., of their complexified Lie algebras \(g^c, \mathfrak{t}, \mathfrak{t}_{M,C}, m^c.\) Let \(b\mapsto b^t\) \((b \in \mathfrak{g})\) denote the unique anti-automorphism of \(\mathfrak{g}\) such that \(X^t=-X\) \((X \in g).\) Consider \(\mathcal{K}\) to be a right \(\mathcal{K}_M\)-module via the multiplication in \(\mathcal{K}:b \circ d=bd\) \((b \in \mathcal{K}, d \in \mathcal{K}_M)\), and consider \(\mathcal{M}\) to be a left \(\mathcal{M}\)-module via the operation \(d \circ c=cd^t\) \((d \in \mathcal{M}, c \in \mathcal{M}).\) We can then form the tensor product \(\mathcal{K} \otimes \mathcal{M}\) of \(\mathcal{K}\)-modules. (We write \(b \otimes c\) for elements of \(\mathcal{K} \otimes \mathcal{M}\), \(b \circ c\) for elements of \(\mathcal{K} \otimes \mathcal{M}).\) The group \(K_M\) acts on \(\mathcal{K} \otimes \mathcal{M}\) via the (well-defined) representation \(\rho: \rho(m)(b \otimes c)=b^m \otimes c^m\) \((b \in \mathcal{K}, c \in \mathcal{M}, m \in K_M).\) Let \((\mathcal{K} \otimes \mathcal{M})^{K_M}\) denote the \(K_M\)-invariants.

**Proposition 1.** \((\mathcal{K} \otimes \mathcal{M})^{K_M}\) is a ring (i.e., the "obvious" multiplication is well defined). In fact, it is a left and right Noetherian integral domain (noncommutative, in general), hence has a quotient division algebra.

We refer to \((\mathcal{K} \otimes \mathcal{M})^{K_M}\) as the C-ring associated to the pair \((P, A).\)

Let \(\tau\) be a left or double representation of \(K\) on a finite-dimensional Hilbert space \(V.\) Then there exists a representation \(\lambda_{\tau}\) of the ring \(\mathcal{K} \otimes \mathcal{M}\) on \(C^\infty(M;V)\) defined as follows:

\[
\lambda_{\tau}(b \otimes c)\psi(m) = \tau(b)\psi(c^m) \quad (b \in \mathcal{K}, c \in \mathcal{M}, m \in M, \psi \in C^\infty(M;V)).
\]

Let \(C^\infty(M, \tau_M)\) denote the space of \(\psi \in C^\infty(M;V)\) such that

\[
\tau(k)\psi(m) = \psi(km) \quad (k \in K_M, m \in M)
\]

if \(\tau\) is a left representation of \(K\) or such that

\[
\tau(k_1)\psi(m)\tau(k_2) = \psi(k_1mk_2) \quad (k_1, k_2 \in K_M, m \in M)
\]
if \( \tau \) is a double representation of \( K \) on \( V \). Then the rule
\[
\lambda_i(\sum b_j \otimes c_j), \psi(m) = \sum \tau(b_j) \psi(c_j m) \quad (b_j \in \mathcal{K}, \ c_j \in \mathcal{M})
\]
defines a representation of the \( C \)-ring \( \mathcal{H} \otimes \mathcal{K} \mathcal{M} \) on \( C^\infty(M, \tau_M) \). Clearly the spaces \( \mathcal{C}(M, \tau_M) \) and \( \mathcal{C}^\infty(M, \tau_M) \) of Schwartz functions and cusp forms in \( C^\infty(M, \tau_M) \) respectively are invariant subspaces.

2. The difference equations satisfied by the \( C \)-function. By a polynomial function on a connected simply connected nilpotent Lie group \( N \), we mean a function \( f \in C^\infty(N) \) such that \( X \mapsto f(\exp X) \ (X \in L(N)) \) is a polynomial function on the Lie algebra \( L(N) \) of \( N \).

By a semilattice \( L \) in a real vector space \( V \), we mean an additive semigroup generated by a basis of \( V \).

**Proposition 2.** There exists a semilattice \( L \subseteq \mathfrak{a}^* \) such that \( \mu \in L \) implies that \( e^{2\mu(H(\mathfrak{a}))} \) is a polynomial function on \( \mathbb{N} \).

**Theorem 1** (The difference equations). Let \( \mu \in \mathfrak{a}^* \) be such that \( e^{2\mu(H(\mathfrak{a}))} \) is a polynomial function on \( \mathbb{N} \). Then there exist polynomials \( b^\mathfrak{a}(\mathfrak{v}) \), \( c^\mathfrak{a}(\mathfrak{v}) \) with coefficients in the \( C \)-ring \( \mathcal{H} \otimes \mathcal{K} \mathcal{M} \) such that, for all double unitary representations \( \tau \) of \( K \),
\[
\lambda_i(b^\mathfrak{a}(\mathfrak{v})) C_{P|P}(1: \mathfrak{v}) = \lambda_i(c^\mathfrak{a}(\mathfrak{v})) C_{P|P}(1: \mathfrak{v} - 2i\mathfrak{\mu}) \quad (\mathfrak{v} \in \mathfrak{a}^*).
\]

The polynomials \( b^\mathfrak{a}(\mathfrak{v}) \) and \( c^\mathfrak{a}(\mathfrak{v}) \) have the same degree and the same leading term, which we may assume lies in \( C[\mathfrak{v}] \) (i.e., is a scalar polynomial). The coefficients of \( c^\mathfrak{a}(\mathfrak{v}) \), in fact, lie in the subring \( \mathcal{Z}_M \) (the center of \( \mathcal{M} \)) of the \( C \)-ring. The operators \( \lambda_i(b^\mathfrak{a}(\mathfrak{v})) \), \( \lambda_i(c^\mathfrak{a}(\mathfrak{v})) \) are never identically zero (as polynomials in \( \mathfrak{v} \)).

Taking \( \mu = \mu_1, \ldots, \mu_t \) to be generators of a semilattice \( L \) as in Proposition 2, we get the result that the \( C \)-function \( C_{P|P}(1: \mathfrak{v}) \) satisfies a system of \( l = rkP \) linear first order partial difference equations with polynomial coefficients.

3. The asymptotic development.

**Theorem 2.** Choose \( \lambda \in \mathfrak{a}^* \) such that \( \Re(\langle \lambda, \ \mathfrak{a} \rangle) > 0 \) for all roots \( \mathfrak{a} \) of the pair \( (P, A) \). Then there exists a formal power series \( \sum_{j=0}^{\infty} t^{-j} b_j^{(\lambda)}(\mathfrak{v}) \) with coefficients in \( (\mathcal{H} \otimes \mathcal{K} \mathcal{M}) \otimes C[\mathfrak{v}] \) (depending analytically on \( \lambda \)) such that
(1) \( b_0^{(\lambda)}(\mathfrak{v}) \in C \);
(2) \( b_j^{(\lambda)}(\mathfrak{v}) \) is of degree at most \( 2j \) in \( \mathfrak{v} \) (\( j \geq 0 \)); and
(3) for every double representation \( \tau \) of \( K \),
\[
C_{P|P}(1: \mathfrak{v} + it\lambda) \sim t^{-s/2} \sum_{j=0}^{\infty} t^{-j} \lambda_i(b_j^{(\lambda)}(\mathfrak{v})) \quad \text{as } t \to \infty
\]
uniformly for \( \nu \) in compact subsets of \( \mathfrak{a}_C^* \) (both sides being considered as operators on the space \( \mathcal{C}(M, \tau_M) \)). This means that, for each integer \( n \geq 0 \),

\[
\lim_{t \to \infty} t^{s/2} C_{P|P}(1; \nu + it\lambda) = \sum_{j=0}^{n} t^{-j} \lambda_j(b_j^{(1)}(\nu)) = 0.
\]

Here \( s = \dim \mathbb{N} \). Replacing \( \tau \) by the trivial representation of \( K \), we get the same asymptotic expansion for the integral \( C_\nu = \int_{\mathbb{R}} e^{it\rho_\nu(\mathfrak{h}(N))} \, d\lambda \).

**Corollary 1.** Suppose that \( \lambda \) is as in Theorem 2. Then there exists a constant \( \zeta_\lambda \) such that

\[
\lim_{t \to \infty} t^{s/2} C_{P|P}(1; \nu + it\lambda) = \zeta_\lambda \times \text{id}
\]

as an operator on \( \mathcal{C}(M, \tau_M) \), the limit being uniform in \( \nu \) on compact subsets of \( \mathfrak{a}_C^* \).

4. The representation theorems.

**Theorem 3.** Choose \( \mu \in \mathfrak{a}^* \) such that \( \langle \mu, \alpha \rangle > 0 \) for all roots \( \alpha \) of \((P, A)\) and \( e^{2\pi i \rho_\nu(\mathfrak{h}(N))} \) is a polynomial function on \( \mathbb{N} \). Let \( b(\nu) = b^\nu(\nu) \), \( c(\nu) = c^\nu(\nu) \) be as in Theorem 1. Then

\[
C_{P|P}(1; \nu) = \text{const} \times \lim_{n \to \infty} n^{-s/2} \lambda_\nu(c(\nu + 2i\mu) \cdots c(\nu + 2in\mu))^{-1} \times \lambda_\nu(b(\nu + 2i\mu) \cdots b(\nu + 2in\mu))
\]

(the constant being independent of \( \tau \)).

**Theorem 4.** Let \((P, A)\) be an arbitrary parabolic subgroup of \( G \); and let \( \tau \) be a double unitary representation of \( K \). Then there exist \( \mu_1, \cdots, \mu_r \in \mathfrak{a}^* \) and constants \( p_{ij}, q_{ij} \) (\( i=1, \cdots, r, j=1, \cdots, j_i \)) depending on \( \tau \) such that

\[
\det C_{P|P}(1; \nu) = \text{const} \times \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma(-i\nu, \alpha_i)/2\langle \mu_i, \alpha_i \rangle + q_{ij}}{\Gamma(-i\nu, \alpha_i)/2\langle \mu_i, \alpha_i \rangle + p_{ij}},
\]

where \( \alpha_1, \cdots, \alpha_r \) are the reduced roots of \((P, A)\).

**Open Question.** Are the numbers \( p_{ij}, q_{ij} \) always rational?

5. Idea of the proofs. Theorem 1 is based on the following sequence of results.

Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) containing \( \mathfrak{a} \); and let \( P_+ \) denote the set of roots \( \beta \) of \((\mathfrak{g}_C, \mathfrak{b}_C)\) such that \( \beta|\mathfrak{a} > 0 \). Let \( X_\beta, X_{-\beta} (\beta \in P_+) \) be root vectors such that \( B(X_\beta, X_{-\beta}) = 1 \) and \( \theta(X_\beta) = -X_{-\beta} = X_{\bar{\beta}} \).

Define vector fields \( q(X) \) (\( X \in \mathfrak{g} \)) on \( \mathbb{N} = \theta(\mathbb{N}) \) by the following rule:

\[
q(X)f(\tilde{n}) = -\sum_{\beta \in P_+} B(X, X_\beta^\tilde{n}) f(\tilde{n}; X_{-\beta}) \quad (f \in \mathcal{C}(\mathbb{N})).
\]
Proposition 3. $X \rightarrow q(X)$ defines a representation of $\mathfrak{g}$ by derivations of $C^\infty(\mathbb{N})$. The ring $\mathcal{R}_G$ of polynomial functions on $\mathbb{N}$ is a $q$-invariant subspace of $C^\infty(\mathbb{N})$.

Let $\sum_\theta (P, A) = \{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of $(P, A)$; and choose $H_j \in a$ such that $\alpha_j(H_j) = \delta_{ij}$.

Proposition 4. Suppose that $X \in \mathfrak{g}$. Then

$$e_\alpha(X) = \left( \sum_{j=1}^l \langle iv - \rho, \alpha_j \rangle B(X, H_j^\alpha) \right) e_\alpha(x) \quad (x \in G).$$

(B is the Killing form on $\mathfrak{g}$; $e_\alpha(x) = e^{iv - \rho(H(x))}$.)

Corollary 2. Suppose that $Z \in \mathfrak{t}$. Then

$$\lambda_i(Z) e_\alpha(\tilde{n}) = \left( \sum_{j=1}^l \langle iv - \rho, \alpha_j \rangle B(Z, H_j^\alpha) \right) e_\alpha(\tilde{n}) \quad (\tilde{n} \in \mathbb{N}).$$

Let $V_1, \cdots, V_t$ ($t = \dim m$) be an orthonormal basis for $m_{C_{\mathbb{N}}}$ (with respect to the Killing form). Also, given $\psi \in C^\infty(M, \tau_{\mathbb{N}})$, define $\check{\Phi} : G \rightarrow C^\infty(M;M)$ by

$$\check{\Phi}(x | m) = \psi(xm) \quad (x \in G, m \in M).$$

Proposition 5. Suppose that $Z \in \mathfrak{t}$ and $\psi \in C^\infty(M, \tau_{\mathbb{N}})$. Then

$$\lambda_i(Z \otimes 1) \check{\Phi}(\tilde{n}) = -q(Z) \check{\Phi}(\tilde{n}) - \sum_j B(Z, V_j) \lambda_i(1 \otimes V_j) \check{\Phi}(\tilde{n}) \quad (\tilde{n} \in \mathbb{N}).$$

Proposition 6. There exists a unique $\mathfrak{h} \otimes \mathbb{C}[v]$ module homomorphism

$$F : \mathfrak{h} \otimes \mathfrak{h} \otimes \mathbb{C}[v] \rightarrow \mathfrak{h} \otimes \mathbb{R} \otimes \mathbb{C}[v]$$

such that

1. $F(1) = 1$;
2. $F(v | \tilde{n})(Z) = \sum_{\alpha} \langle iv + \rho, \alpha_j \rangle B(Z, H_j^\alpha) - \sum_j B(Z, V_j) V_j$ $(Z \in \mathfrak{t})$;
3. $F(Zb) = F(b)F(Z) + q(Z)F(b)$ $(Z \in \mathfrak{t}, b \in \mathfrak{h})$;
4. $F(b \otimes c) = cF(b)$ $(b \in \mathfrak{h}, c \in \mathfrak{h})$.

Proposition 7. Suppose that $b \in \mathfrak{h} \otimes \mathfrak{h} \otimes \mathbb{C}[v]$. Then there exists a constant $C = C(b) > 0$ such that if $\Im \langle v, \alpha_j \rangle \geq C(b)$ ($j = 1, \cdots, l$), then

$$\lambda_i(b) \int_S e_\alpha(\tilde{n}) \psi(\tilde{n}m) d\tilde{n} = \int_S e_\alpha(\tilde{n}) \lambda_i(1 \otimes F(v | \tilde{n})(b)) \check{\Phi}(\tilde{n} | m) d\tilde{n} \quad (m \in \mathbb{M}),$$

both integrals being convergent.

Proposition 8. Given $\phi(\tilde{n}) \in \mathcal{R}_G$, we can find $b(v) \in \mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h} \otimes \mathbb{C}[v]$ and $c(v) \in \mathfrak{h} \otimes \mathbb{R} \otimes \mathbb{C}[v]$ such that $F(b(v)) = c(v) \phi$. [September
Proof of Theorem 1. First apply Proposition 8 with \( \phi(\bar{n}) = e^{2\mu(H(\bar{n}))} \). Then apply Proposition 7.

The following is the essential step in the proof of Theorem 2.

**Proposition 9.** Suppose that \( \nu \in \mathfrak{a}_C^* \) and let \( f_\nu(\bar{n}) = \nu(H(\bar{n})) \) (\( \bar{n} \in \bar{N} \)). Then if \( \langle \nu, \alpha \rangle \neq 0 \) for all \( \alpha \in \sum (P, A) \), \( \bar{n} = e \) is the only critical point of \( f_\nu \), and \( \bar{n} = e \) is a nondegenerate critical point. Furthermore if \( \nu \in \mathfrak{a}_C^* \) and \( \langle \nu, \alpha \rangle > 0 \) for all \( \alpha \in \sum (P, A) \), then the critical point of the (real-valued) function \( f_\nu(\bar{n}) \) has index 0.

Proposition 9 allows us to apply the method of steepest descent (see [1]) to derive the asymptotic expansion of \( C_{P|P}(1: \nu) \) (Theorem 2).

Theorems 3 and 4 follow fairly easily, given Theorems 1 and 2.

6. An example: the C-function for the group SU(1, 2). In this case, the set \( P_+ \) consists of three roots \( \beta_1, \beta_2, \beta_3 \), where \( \beta_1 \) and \( \beta_2 \) are simple and \( \beta_3 = \beta_1 + \beta_2 \). Also, the parabolic pair \( (P, A) \) is minimal; so the C-ring is isomorphic to \( \mathfrak{K}^* \). If \( \mu = \alpha \) (the simple root of \( (P, A) \)), \( e^{2\mu(H(\bar{n}))} \) is a polynomial function on \( \bar{N} \); the corresponding polynomials \( b^\mu(\nu) \) and \( c^\mu(\nu) \) are then as follows

\[
\begin{align*}
 b^\mu(\nu) &= b_1^\mu(\nu)b_2^\mu(\nu), \\
 c^\mu(\nu) &= c_1^\mu(\nu)c_2^\mu(\nu),
\end{align*}
\]

where

\[
\begin{align*}
 b_1^\mu(\nu) &= \{(i\nu + \rho, \alpha) - i(\sqrt{6}(6)Z_{\beta_3})(i\nu, \alpha) + \frac{1}{2} V + \frac{1}{3} Z_{\beta_3}Z_{\beta_3}^*,
 b_2^\mu(\nu) &= \{(i\nu + \rho, \alpha) - \frac{1}{2} V)(i\nu, \alpha) + i(\sqrt{6}(6)Z_{\beta_3}) + \frac{1}{3} Z_{\beta_3}Z_{\beta_3},
\end{align*}
\]

and

\[
 c^\mu(\nu) = (i\nu + \rho, \alpha)(i\nu + \alpha, \nu)(i\nu + \alpha) + \frac{1}{2} V)(i\nu + \rho, \alpha) - \frac{1}{2} V).
\]

Here \( Z_{\beta_i} = \frac{1}{i}(X_{\beta_i} + \theta(X_{\beta_i})) \) (\( X_{\beta_i} \) normalized as above), and \( V \) is the element of \( \mathfrak{m}_C \) such that \( \beta_1(V) = \frac{1}{3} \).

Using the polynomials \( b^\mu(\nu) \) and \( c^\mu(\nu) \), we obtain the following result.

**Proposition 10.** Let \( \tau = \tau_{m, n} \) be the \((m+1)\)-dimensional representation of \( K = U(2) \) such that \( \tau(V - i\sqrt{6}Z_{\beta_3}) = n \times 1 \). \((V - i\sqrt{6}Z_{\beta_3} \) spans the center of \( \mathfrak{g}_C \).) Let \( \mathcal{V} = \mathcal{V}^{(m, n)} \) denote the space of \( \tau \). Then \( \mathcal{V} \) has a basis \( \chi_j \) \((j = 0, 1, \cdots, m) \) such that \( \tau(V)\chi_j = \frac{1}{4}(m+n-2j)\chi_j \). Furthermore, the operator

\[
 C^{(m, n)}_{P|P}(1: \nu) = \int_{\bar{R}} \tau(k(\bar{n}))e^{i\nu - \rho(H(\bar{n}))} \, d\bar{n}
\]

on \( \mathcal{V}^{(m, n)} \) has each vector \( \chi_j \) as an eigenvector; and the corresponding eigenvalue is

\[
 2 \frac{\Gamma(\zeta_3)\Gamma(\zeta_5)\Gamma(\zeta_6)\Gamma(\zeta_4)}{\sqrt{\pi}\, \Gamma(\zeta_5)\Gamma(\zeta_6)\Gamma(\zeta_7)\Gamma(\zeta_8)}
\]
where \( \zeta_1 = \zeta = -i \langle \nu, \alpha \rangle /2 \langle \alpha, \alpha \rangle, \) \( \zeta_2 = \zeta + \frac{1}{2}, \) \( \zeta_3 = \zeta + \frac{3}{2}j - \frac{3}{2}m - \frac{3}{2}n, \) \( \zeta_4 = \zeta - \frac{3}{2}j + \frac{3}{2}m + \frac{3}{2}n, \) \( \zeta_5 = \zeta + \frac{1}{2}j - \frac{3}{2}m - \frac{3}{2}n, \) \( \zeta_6 = \zeta + \frac{1}{2}j + \frac{3}{2}m - \frac{3}{2}n + 1, \) \( \zeta_7 = \zeta - \frac{1}{2}j - \frac{3}{2}m + \frac{3}{2}n, \) and \( \zeta_8 = \zeta - \frac{1}{2}j + \frac{3}{2}m + \frac{3}{2}n + 1. \)

Detailed proofs of these results and some more examples will appear in a paper in preparation.

REFERENCES


