AN EMBEDDING-OBJECTION FOR PROJECTIVE VARIETIES

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A classical problem in differential topology is the following: Let $X$ be a compact $n$-dimensional differentiable manifold (without boundary). Then compute the least integer $m=m(X)$ such that $X$ may be embedded into $\mathbb{R}^m$. Usually this question is attacked as follows (see Atiyah [1]): (a) An upper bound for $m$ is obtained by exhibiting explicit embeddings, and (b) a lower bound is obtained by certain homotopy invariants.

The forthcoming paper [2] deals with an algebro-geometric counterpart to the problem mentioned above: Let $X$ be a nonsingular, projective $k$-variety embedded in some projective space $\mathbb{P}^N_k$ by the embedding $i$. For simplicity we assume the field $k$ to be algebraically closed, but the results of [2] still hold under the weaker assumption that $k$ is infinite.

The main result is that the least integer $m=m(X,i)$, such that $X$ can be embedded into $\mathbb{P}^m_k$ via a projection from $\mathbb{P}^N_k$, is effectively computed in terms of the degrees of the Chern-classes of $X$.

More precisely, let $X \subset \mathbb{P}^N_k$ be an $n$-dimensional nonsingular projective variety, embedded in $\mathbb{P}^N_k$. Let $c_i=c_i(X)=c_i(\mathcal{O}_{X/k}) \in A(X)$ be the Chern-classes of $X$, where $A(X)$ denotes the Chow-ring of $X$. Consider the formal inverse of the alternating Chern-polynomial:

$$\left[ \sum_{i=0}^{n} (-1)^i c_i T^i \right]^{-1} = \sum_{i=0}^\infty f_i T^i.$$

Here $f_i=0$ for $i>n$. Let $d_i=\deg(f_i)$ with respect to the embedding $i:X \subset \mathbb{P}^N_k$. In particular $d_0=\deg(i(X))=d$. Define

$$B_X(T) = \left( \sum_{i=0}^{n} d_i T^i \right) \left( \sum_{i=0}^{2n+1} \binom{2n+2}{i} T^i \right) = B_0 + B_1 T + \cdots,$$


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of which we only need $B_0, B_1, \cdots, B_n$. In fact, we put
\[
\beta_j = \sum_{i=0}^{j-n} (-1)^i \binom{n-i}{j-i-n} \left(B_i - d^n_i \binom{n+1}{i}\right), \quad n \leq j \leq 2n^2.
\]
\[
\beta_j = 1 \quad \text{for } j < n, \quad \beta_j = 0 \quad \text{for } j > 2n.
\]

**Definition.** For all integers $m$ the sequence $(\beta_m, \beta_{m+1}, \cdots)$ is called the $m$th embedding obstruction of the embedded variety $(X, i)$.

In [2] the following result is proved:

**Theorem.** If $m < N$, then $X$ can be embedded into $\mathbb{P}^m_k$ via a projection from $\mathbb{P}^N_k$ if and only if the $m$th obstruction vanishes, i.e.

\[
(\beta_m, \beta_{m+1}, \cdots) = (0, 0, \cdots).
\]

This implies at once the well-known and classical (see E. Lluis [5]):

**Corollary.** $m(X, i) \leq 2n+1$.

For $n=1$ and $m=2$ we obtain the well-known genus-formula

\[
g(X) = \frac{1}{2}(d - 1)(d - 2)
\]

which is necessary and sufficient for when the nonsingular curve $X$ can be projected isomorphically onto a plane curve. For $n=2, m=3$, we get that a nonsingular surface $X$ in $\mathbb{P}^N_k$ can be embedded into $\mathbb{P}^3_k$ via a projection if and only if

\[
\deg(K_X) = (d - 4)d,
\]

\[
(K_X^2) = (d - 4)^2d,
\]

\[
p_a(X) = \frac{1}{6}(d - 1)(d - 2)(d - 3).
\]

Again $d = \deg(X)$, $K_X$ is the canonical divisor and $p_a(X)$ the arithmetic genus of $X$. The necessity of (3) was noted by Iversen [4].

It should be easy to compute formulas similar to (2) and (3) in any dimension $n$ by means of (1), and thus obtain a characterization (in terms of classical invariants like $K_X, p_a(X)$) of those nonsingular varieties $X$ in $\mathbb{P}^N_k$ which can be projected isomorphically onto a hypersurface in $\mathbb{P}^{n+1}_k$.

Of course (1) with $m = n+1$ gives such a characterization, in terms of the degrees of certain monomials in the Chern-classes of $X$.

Another application of the theorem is to Abelian varieties. In fact, the question of embeddings for Abelian varieties is resolved as follows: Let $X \subseteq \mathbb{P}^N_k$ be an $n$-dimensional Abelian variety. Then:

(i) $X$ can always be embedded into $\mathbb{P}^{2n+1}_k$ via a projection from $\mathbb{P}^N_k$;

(ii) $X$ can be embedded into $\mathbb{P}^{2n}_k$ via a projection from $\mathbb{P}^N_k \iff \deg(X) = \frac{1}{4}(2n+1)$;

(iii) $X$ cannot be embedded into $\mathbb{P}^{2n-1}_k$.

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\[\text{ADDED IN PROOF. Using standard combinatorial identities, one easily checks that}\]

\[-1)^{-n} \beta_i = \left(\sum_{i=0}^{j-n} (-1)^{j-i-n} d_i \right) - d^n_i.
\]
For $n=1$, (ii) gives $\deg(X)=3$ which is no surprise, and for $n=2$ we get $\deg(X)=10$. The necessity of this condition for the embedding of a 2-dimensional Abelian variety into $\mathbb{P}^4_\mathbb{C}$ was noted by Horrocks and Mumford in [3, Theorems 5.1 and 5.2].

It should be noted that [2] deals only with embedded projective varieties. For a given projective variety $X$, one may ask for the least integer $e=e(X)$ such that $X$ may be embedded into $\mathbb{P}^e_\mathbb{C}$. If $X$ is given as a subvariety of some $\mathbb{P}^n_\mathbb{C}$, one may very well have $m(X)>e(X)$. Nevertheless, calculation of $m(X)$ can be used to obtain upper and lower bounds for $e(X)$, see for example the computation for Abelian varieties referred to above. In order to compute $e(X)$, one must find the projective embeddings $i$ of $X$ for which $m(X,i)$ is minimal, i.e., for which the embedding obstruction is as nice as possible. We hope to return to this question later.

References


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