ON THE EXTENSION OF BOUNDARY INTEGRABLE ALMOST COMPLEX STRUCTURE

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1. Introduction. Let \{M, M'\} be a finite Kähler manifold, i.e., \(M'\) is a complex Kähler manifold, \(M\) is an open submanifold of \(M'\) with compact closure, \(M_0 = \partial M\), the boundary of \(M\), is a \(C^\infty\) submanifold of \(M'\), and for each \(p \in M_0\) there exists a coordinate neighborhood \(U\) of \(p\) with real coordinates \(t_1, \cdots, t_{2n-1}\), \(r\) such that \(r(q) < 0\) for \(q \in U \cap M\) and \(r(q) > 0\) for \(q \in U \cap (M' - M)\). It is assumed that the following conditions hold:

A. For each boundary point the Levi form has at least two positive eigenvalues.

B. There exists a constant \(c_0 > 0\) such that for all \(u \in C^0, q(M, 0), \# = 1, 2 \ (\langle 2\Box - \Delta \rangle u, u) \geq c_0(u, u)\) where \(\Theta\) is the holomorphic tangent bundle of \(M'\), \(C^p, q(M, 0)\) is the space of all \(C^\infty\) \(\Theta\)-valued \((p, q)\)-forms extendible to a neighborhood of \(M\), \(\Box\) (resp., \(\Delta\)) is the complex (resp., the real) Laplacian on \(C^p, q(M, 0)\) and \(\langle , \rangle\) is the \(L^2\)-inner product over \(M\) (see [2]).

Then the main result of this note states that a sufficiently small integrable almost-complex structure on \(M_0\) can be extended to a complex structure on \(M\). A complete proof will appear elsewhere; a brief outline follows.

However, we first take a closer look at condition B. Let \(D\) be the covariant differentiation operator associated with the connection \(\theta\) of the metric \(g\) on \(M'\), i.e.,

\[ Du = d\mu + \theta \wedge u = \delta u + \bar{\delta} u \]

for \(u \in C^p, q(M, \Theta)\). Let \(D^*\) and \(\delta^*\) be the formal adjoints of \(D\) and \(\delta\), respectively. Then \(\Delta = DD^* + D^*D\) and \(\Box = \delta^*\delta + \delta\delta^*\). Since \(g\) is Kähler, \(\Delta = 2\Box - K, K = \sqrt{-1}(e(s)\Lambda - \Lambda e(s))\), where

\[ e(s)u = \delta^* u, \quad \Lambda u = *^{-1}(\rho \wedge * u), \]

\(*\) is the Hodge star operator and \(\rho\) is the Kähler form of \(g\). We refer


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982
to [3, pp. 482–483], for verification of this identity. Hence, condition B requires the existence of a constant $c_0 > 0$ such that $(Ku, u) \geq c_0(u, u)$ for all $u \in C^{0,q}(\mathcal{M}, \Theta)$, $q = 1, 2$. Now it is established in [2, p. 276], that if the scalar curvature is sufficiently negative, then one has the stronger result $\langle Ku, u \rangle_2 \geq c_0(u, u)_2$ for all $x \in \mathcal{M}'$, where $\langle \cdot, \cdot \rangle_2$ is the inner product at the point $x$, i.e., $\Theta$ is $W^{0,q}$-elliptic. It is also shown in [2] that the criterion of $W$-ellipticity is satisfied for a large class of bounded homogeneous domains in $\mathbb{C}^n$ provided with the Bergman metric. More generally, let $\mathcal{M}'$ be a manifold whose universal covering space $\tilde{\mathcal{M}}'$ is isomorphic to $D_1 \times \cdots \times D_r$, where $D_i$ is a bounded irreducible symmetric domain with $\dim_{\mathbb{C}} D_i \geq 3$. Then $\Theta$ is $W^{0,q}$-elliptic for $0 \leq q \leq 2$, and condition B will hold for any relatively compact open submanifold $M$ of $\mathcal{M}'$ with smooth boundary.

2. Definitions and notation. Let $M_0$ be a $C^\infty$ manifold of real dimension $2n-1$ and let $CTM_0$ be the complexified tangent bundle.

2.1. DEFINITION. An almost-complex structure on $M_0$ is given by a complex subbundle $E''$ of $CTM_0$ of fiber complex dimension $n-1$ such that $E'' \cap E'' = \{0\}$.

2.2. DEFINITION. The almost-complex structure $E''$ on $M_0$ is integrable if, for any two sections $L$ and $L'$ of $E''$ over an open set $U$ of $M_0$, $[L, L']$ is also a section of $E''$.

We now assume that $M_0$ is the boundary of a finite complex manifold $(\mathcal{M}, M')$. The complex structure on $M'$ induces an integrable almost-complex structure $T''$ on $M_0$.

2.3. DEFINITION. The almost-complex structure $E''$ on $M_0$ is of finite distance from $T''$ if $\pi''|E'': E'' \to T''$ is an isomorphism where $\pi'' : CTM_0 \to T''$ is the projection.

In this case $E'' = \{X - \tau \circ \varphi(X) | X \in T''\}$ where $\tau : \Theta|\mathcal{M}_0 \to T'' \oplus CF$ is an isomorphism, $T' = T''$, $CF$ is the complexification of a real one-dimensional subbundle $F$ of $TM_0$ such that $CTM_0 = T'' \oplus T'' \oplus CF$ and

$$\varphi = -\tau^{-1} \circ (\text{id} - \pi'') \circ (\pi''E'')^{-1} : T'' \to \Theta \mid M_0,$$

i.e., $\varphi$ is a $\Theta|M_0$-valued $C^{\infty}$ differential form on $M_0$ of type $(0, 1)$. Conversely, any such differential form $\varphi$ gives rise to an almost complex structure $E''$ on $M_0$. We will denote $E''$ by $T''_\omega$. As in the case of complex manifolds, there exists a $\Theta|\mathcal{M}_0$-valued $C^{\infty}$ differential form $\Phi$ on $M_0$ of type $(0, 2)$, such that $\Phi = 0$ if and only if $T''_\omega$ is integrable.

Let $\varphi$ be a $T'$-valued form and let $\omega \in C^{0,1}(\tilde{\mathcal{M}}, \Theta)$ be such that $t\omega$, the complex tangential part of $\omega$, is equal to $\varphi$ on $M_0$. Let $\Omega = \tilde{\partial}\omega - [\omega, \omega]$. If $T''_\omega$ is the almost complex structure on $M$ induced by $\omega$, then one can show that $T''_\omega = CTM_0 \cap T''_\omega$ and $t\Omega = 0$ on $M_0$ if and only if $\Phi = 0$. 

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3. The main result. Now we can formulate the following extension problem.

**Theorem.** Let \( \{M, M'\} \) be a finite complex Kähler manifold such that conditions A and B in §1 are satisfied. Let \( \varphi \) be a \( T' \)-valued \( C^\infty \) differential form of type \((0, 1)\) with sufficiently small Hölder norm \( |\varphi|_{k+\alpha} \), \( 0 < \alpha < 1 \), for some integer \( k > 0 \) depending on \( n \). Assume that \( T'_\varphi \) is integrable. Then there exists \( \omega \in C^{0,1}(\overline{M}, \Theta) \) such that \( \Omega = 0 \) and \( \omega = \varphi \) on \( M_\varphi \).

We first consider the quadratic form

\[
Q(u, v) = \frac{1}{2}[(Du, Dv) + (D^*u, D^*v) + (Ku, v)] - 2([\varphi, u], \delta v)
\]

for some \( \varphi \in C^{0,1}(\overline{M}, \Theta) \) with sufficiently small norm and \( u, v \in \mathcal{B} = \{\omega \in C^{0,1}(\overline{M}, \Theta)| \omega_0 = 0 \text{ on } M_\varphi \} \). One can easily check that by condition B, \( \text{const} \ N^2(u) \leq |\text{Re } Q(u, u)| \leq \text{const} \ N^2(u) \) where \( \text{Re} \) stands for the real part of \( Q(u, u) \), and \( N^2(u) = \|u\|_g^2 + \|Du\|_g^2 + \|D^*u\|_g^2 \). Hence, if \( \|u\|_g \) is the Sobolev \( s \)-norm of \( u \), then \( \|u\|_1 \leq \text{const}|\text{Re } Q(u, u)| \).

It follows from the theory developed in [1] and [4] that for each \( \sigma \in C^{0,1}(\overline{M}, \Theta) \) there exists a unique \( u \in \mathcal{B} \) such that \( Q(u, v) = (\sigma, v) \) for all \( v \in \mathcal{B} \), the completion of \( \mathcal{B} \) with respect to the norm \( N \) such that

\begin{align*}
(1) & \quad \|u\|_{s+2} \leq c_s \|\sigma\|_s; \\
(2) & \quad L_\varphi u = \frac{1}{2}(DD^* + D^*D + K)u - 2\delta^*[\varphi, u] = \sigma; \\
(3) & \quad tD^*u = 0 \quad \text{on } M_\varphi; \\
(4) & \quad |u|_{k+\alpha+2} \leq c_k \|f\|_{k+\alpha}
\end{align*}

for sufficiently large \( k \). The constants \( c_s \) and \( c_k \) depend on \( s \) and \( k \) and on the derivatives of \( \varphi \) up to order \( s \) and \( k \), respectively. If \( |\varphi|_{k+\alpha} \) is sufficiently small we may assume that \( c_k \) in (4) depends only on \( k \).

We observe that \( D^*u = -\delta \ast u - \delta \ast u \), and since \( u \) is a form of type \((0, 1)\), \( \delta \ast u = 0 \) and \( D^*u = \delta^*u \). On the other hand for a Kähler metric \( g \) the complex Laplacian \( \Box = \delta^5 \ast + \delta \ast \delta \) is \( \frac{1}{2}(DD^* + D^*D + K) \), and if \( \sigma = \delta^*h \) for \( h \in C^{0,2}(\overline{M}, \Theta) \), then (3) and Stokes' theorem imply that \( L_\varphi u = \sigma \) if and only if

\[
\delta^* \delta u - 2\delta^*[\varphi, u] = \delta^*h.
\]

We now consider the nonlinear differential system \( \delta^*\Omega = 0 \). Let \( \omega_0 \in C^{0,1}(\overline{M}, \Theta) \) be an extension of \( \varphi \) such that \( |\omega_0|_{k+\alpha} \leq \text{const}|\varphi|_{k+\alpha} \). One can inductively construct a sequence of approximate solutions \( \omega_{m+1} = \omega_m + u_m \), where \( u_m \) is the solution of (5) with \( tu_m = \delta^*u_m = 0 \) on \( M_\varphi \), \( \psi = \omega_m \), \( h = -\Omega_m = -\delta \omega_m + |\omega_m, \omega_m| \). Since \( |\delta^*\Omega_m|_{k+\alpha} \leq \text{const}|u_m|_{k+\alpha} \), (4) implies that there exists a constant \( c > 0 \) such that \( |\omega_{m+1} - \omega_m|_{k+\alpha} \leq c|\omega_m - \omega_{m-1}|_{k+\alpha} \).
for $m=1, 2, \cdots$. This is enough to conclude that there exists a $\Theta$-valued form $\omega$ of type $(0, 1)$ and of class $C^{k+\alpha}$ on $M$ such that $\delta^*\Omega=0$, $t\omega=\varphi$ on $M_0$, and $|\omega|_{k+\alpha} \leq \text{const} |\varphi|_{k+\alpha}$.

Now it can easily be shown that $\delta\Omega=2[\omega, \Omega]$. By condition A and the fact that the normal part of $\ast \# \Omega$ vanishes on $M_0$, the basic estimate of the $\delta$-Neumann problem holds for $\ast \# \Omega$, i.e.,

$$E(* \# \Omega) \leq \text{const} (\|\Omega\|^2 + \|\delta \Omega\|^2 + \|\delta^* \Omega\|^2).$$

For the definition of the norm $E$, we refer to [5] and [6]; the operators $*$ and $\#$ are defined in [2]. Then by condition B and the complete continuity of $E$, one can obtain the estimate $\|\delta \Omega\| \leq c_0 |\omega|_{1,\alpha}$ for some constant $c_0$. Thus $\delta \Omega=0$ if $|\varphi|_{k+\alpha}$ is sufficiently small. Since $t\Omega=0$ on $M_0$, $\delta \Omega=0$, and $\delta^* \Omega=0$, condition B implies that $\Omega=0$.

Finally, it follows from the construction of approximate solutions that $\omega=\omega_0+w$, where $w$ is of class $C^{k+\alpha}$ and $\delta^* w=0$. Then $\delta^*(\delta \omega-[\omega, \omega])=0$ can be expressed as

$$\Box w = \delta^* (2[\omega_0, w] + [w, w]) = \delta^* ([\omega_0, \omega_0] - \delta \omega_0).$$

This equation is elliptic if $|\varphi|_{k+\alpha}$ is sufficiently small. Since $\omega_0$ is of class $C^\infty$, $w$ is also of class $C^\infty$.

REFERENCES


