UNBOUNDED OPERATORS WITH SPECTRAL CAPACITIES

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Let \( \mathcal{S}(X) \) denote the family of subspaces (closed linear manifolds) of a Banach space \( X \), and let \( \mathcal{F} \) and \( \mathcal{R} \) represent the collection of closed and compact subsets of the complex plane \( \pi \), respectively. The superscript \( c \) stands for the complement.

1. Definition [1]. A spectral capacity in \( X \) is an application \( \mathcal{E}: \mathcal{F} \to \mathcal{S}(X) \) which satisfies the following conditions:

   (i) \( \mathcal{E}(\emptyset) = \{0\}, \mathcal{E}(\pi) = X \);

   (ii) \( \bigcap_{n=1}^{\infty} \mathcal{E}(F_n) = \mathcal{E}(\bigcap_{n=1}^{\infty} F_n), \{F_n\} \subset \mathcal{F} \);

   (iii) for every finite open cover \( \{G_i\}_{i=1}^{m} \) of \( F \in \mathcal{F} \), \( \mathcal{E}(F) = \bigcup_{i=1}^{m} \mathcal{E}(F \cap G_i) \).

In order to confine the present investigation to densely defined operators on \( X \), the following additional constraint on the spectral capacity is needed:

2. Definition. A spectral capacity \( \mathcal{E} \) will be referred to as regular if the linear manifold \( X_0 = \{x \in \mathcal{E}(K): K \in \mathcal{R}\} \) is dense in \( X \).

3. Definition. A linear operator \( T: D(T) (\subseteq X) \to X \) is said to possess a regular spectral capacity \( \mathcal{E} \) (abbrev. \( T \in \mathcal{I}(\mathcal{E}) \)) if it is closed, has a nonvoid resolvent set and satisfies the following conditions:

   (iv) \( \mathcal{E}(K) \subseteq D(T) \) for all \( K \in \mathcal{R} \);

   (v) \( T(\mathcal{E}(F) \cap D(T)) \subseteq \mathcal{E}(F) \) for all \( F \in \mathcal{F} \);

   (vi) the restriction \( T_F = T|_{\mathcal{E}(F) \cap D(T)} \) has the spectrum \( \sigma(T_F) \subseteq F, F \in \mathcal{F} \).

4. Theorem. Given \( T \in \mathcal{I}(\mathcal{E}) \). For every \( K \in \mathcal{R} \), the restriction \( T_K = T|_{\mathcal{E}(K)} \) is a (bounded) decomposable operator on \( \mathcal{E}(K) \) possessing the

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spectral capacity $\mathcal{E}_K$ defined by

$$
(1) \quad \mathcal{E}_K(F) = \mathcal{E}(K \cap F) \quad \text{for all } F \in \mathcal{F}.
$$

In the proof it is shown that $T_K$ is bounded by the closed graph theorem and $\mathcal{E}_K$, as defined by (1), is a spectral capacity for $T_K$.

A property which is instrumental for the subsequent study of operators in $\mathcal{F}(\mathcal{E})$ is expressed by the following

5. Theorem. Let $T \in \mathcal{L}(\mathcal{E})$ and $K \in \mathcal{R}$. The following statements are equivalent:

(i) $x \in \mathcal{E}(K)$;

(ii) there exists an $X$-valued function $\hat{x}$ analytic on $K^c$ satisfying the equation

$$
(\lambda - T)\hat{x}(\lambda) = x \quad \text{for all } \lambda \in K^c.
$$

The implication (i) $\Rightarrow$ (ii) of the proof is based on the single-valued extension property of a decomposable operator. (ii) $\Rightarrow$ (i) is proved first for an $x \in X_0$ with the help of a result by C. Foiaş [4]:

$$
\{y \in \mathcal{E}(L): \sigma_{T_L}(y) \subseteq K\} = \mathcal{E}(K) \quad \text{where } L(\supset K) \in \mathcal{R}.
$$

Next, for $x \notin X_0$, the density of $X_0$ in $X$ and the closeness of $\mathcal{E}(K)$ complete the proof.

6. Theorem. Every $T \in \mathcal{L}(\mathcal{E})$ has a unique regular spectral capacity.

In the first stage of the proof, the application of Theorem 5 shows that any two regular spectral capacities $\mathcal{E}$ and $\mathcal{E}_1$ of $T$ agree on $\mathcal{R}$. Next the property expressed by Definition 2 implies that $\mathcal{E}(F) = \mathcal{E}_1(F)$ for all $F \in \mathcal{F}$.

7. Theorem. For every $K \in \mathcal{R}$, $\mathcal{E}(K)$ is a spectral maximal space of $T \in \mathcal{L}(\mathcal{E})$.

The proof is performed with the help of Theorems 4 and 5.

8. Theorem. Given $T \in \mathcal{L}(\mathcal{E})$. For every $x \in X$ there exists a nonvoid open set $U \subseteq \pi$ and a sequence $\{\hat{x}_n\}$ of $X$-valued functions analytic on $U$, with

$$
\lim_{n \to \infty} (\lambda - T)\hat{x}_n(\lambda) = x \quad \text{for all } \lambda \in U.
$$

Again, the proof is obtained by an application of Theorem 5.

We redefine E. Bishop's concept of weak spectral manifold $\mathcal{M}(F, T)$ [2, Definition 2] without the restriction of $T$ being bounded as follows: Given $T \in \mathcal{D}(T)$ ($\subseteq X$) $\rightarrow X$ and $F \in \mathcal{F}$, $\mathcal{M}(F, T)$ is the set of all $x \in X$ which
have the property that for each $\varepsilon > 0$ there exists an $X$-valued function $\tilde{x}$ analytic on $F^c$ such that $\|x - (\lambda - T)\tilde{x}(\lambda)\| < \varepsilon$, for all $\lambda \in F^c$.

A straightforward consequence of Theorem 8 is the following

9. Corollary. Given $T \in \mathcal{F}(E)$. For every $F \in \mathcal{F}$,

$$\mathcal{E}(F) = \mathfrak{N}(F, T).$$

REFERENCES


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