A FIXED POINT THEOREM FOR MULTIVALUED NONEXPANSIVE MAPPINGS IN A UNIFORMLY CONVEX BANACH SPACE

BY TECK-CHEONG LIM

Communicated by Alberto Calderón, February 19, 1974

Let $C$ be a nonempty weakly compact convex subset of a Banach space $X$, and $\mathcal{C}(C)$ be the family of nonempty compact subsets of $C$ equipped with the Hausdorff metric. Let $T: C \rightarrow \mathcal{C}(C)$ be a nonexpansive mapping, i.e. for each $x, y \in C$,

$$H(T(x), T(y)) \leq \|x - y\|,$$

where $H(A, B)$ denotes the Hausdorff distance between $A$ and $B$. A point $x \in C$ is called a fixed point of $T$ if $x \in Tx$. Fixed point theorems for such mappings $T$ have been established by Markin [11] for Hilbert spaces, by Browder [2] for spaces having weakly continuous duality mapping, and by Lami Dozo [7] for spaces satisfying Opial’s condition. Lami Dozo’s result is also generalized by Assad and Kirk [1]. By making use of Edelstein’s asymptotic center [4], [5], we are able to prove Theorem 1. Let $C$ be a closed convex subset of a uniformly convex Banach space and let $\{u_i\}$ be a bounded sequence in $C$. The asymptotic center $x$ of $\{u_i\}$ in (or with respect to) $C$ is the unique point in $C$ such that

$$\limsup_i \|x - u_i\| = \inf \left\{ \limsup_i \|y - u_i\| : y \in C \right\}.$$

The number $r = \inf \{\limsup_i \|y - u_i\| : y \in C \}$ is called the asymptotic radius of $\{u_i\}$ in $C$. Existence of the unique asymptotic center is proved by Edelstein in [5]. Results on ordinal numbers used here may be found in [13].

**Theorem 1.** Let $X$ be a uniformly convex Banach space and $C$ be a closed convex bounded nonempty subset of $X$. Let $T: C \rightarrow \mathcal{C}(C)$ be a nonexpansive mapping from $C$ into the family of nonempty compact subsets of


*Key words and phrases.* Fixed point, multivalued nonexpansive mapping, uniformly convex Banach space, asymptotic center.

1 This research was conducted while the author held an Izzak Walton Killam Memorial Scholarship under the supervision of Professor Michael Edelstein.
C (equipped with the Hausdorff metric). Then T has a fixed point, i.e. there exists \( x \in C \) with \( x \in Tx \).

**Proof.** Let \( a \) be a point in \( C \) fixed throughout the proof. Let \( \{ \lambda_m \} \) be a decreasing sequence of positive numbers and \( \lim \lambda_m = 0 \). For each \( m \), the mapping \( T_m : C \to C \) defined by \( T_m(x) = \lambda_m a + (1 - \lambda_m) Tx \) is a contraction mapping and hence has a fixed point \( x_m \) (Nadler [12]). Thus \( x_m \in \lambda_m a + (1 - \lambda_m) Tx_m \), and there exists \( y_m \in Tx_m \) with \( x_m = \lambda_m a + (1 - \lambda_m) y_m \). Since \( C \) is bounded, we have

\[
\|x_m - y_m\| = \lambda_m \|a - y_m\| \to 0 \quad \text{as } m \to \infty.
\]

To facilitate the later description, we define \( i : \{x_m\} \to \{y_m\} \) by \( i(x_m) = y_m \) for all \( m \). We say that a sequence \( \{x_n\} \) is an essential subsequence of \( \{y_n\} \) if for some \( N > 0 \), \( \{x_n\}_{n \geq N} \) is a subsequence of \( \{y_n\} \).

Define the sequence \( \{x_m^{(\delta)}\} \) to be \( \{x_m\} \), i.e. \( x_m^{(\delta)} = x_m \) for each \( m \geq 1 \). Let \( \Omega \) be the first uncountable ordinal and \( \beta \) be a countable ordinal, i.e. \( \beta < \Omega \).

Suppose that \( \{x_m^{(\delta)}\} \) has been defined for every ordinal \( \alpha \) less than \( \beta \) in such a way that \( \{x_m^{(\delta)}\} \) is an essential subsequence of \( \{x_n^{(\gamma)}\} \) whenever \( \delta < \gamma < \beta \).

We define \( \{x_m^{(\beta)}\} \) as follows:

**Case 1.** \( \beta \) has an immediate predecessor, i.e., \( \beta = \alpha + 1 \) for some \( \alpha < \Omega \). Let \( z_\alpha \) be the asymptotic center of \( \{x_m^{(\alpha)}\} \) in \( C \). For each \( m \), let \( p_m \in Tz_\alpha \) be chosen such that

\[
\|p_m - y_m^{(\alpha)}\| \leq \|z_\alpha - x_m^{(\alpha)}\|
\]

where \( y_m^{(\alpha)} = i(x_m^{(\alpha)}) \); existence of such a \( p_m \) is a consequence of the non-expansiveness of \( T \) and the compactness of \( Tz_\alpha \). Since \( Tz_\alpha \) is compact and \( \{p_m\} \subseteq Tz_\alpha \), there exists a convergent subsequence \( \{p_m\} \) of \( \{p_m\} \). We then define \( \{x_m^{(\beta)}\} \) to be the sequence \( \{x_m^{(\beta)}\} \).

**Case 2.** \( \beta \) is a limit ordinal. Then there exists a strictly increasing sequence \( \{\alpha_n\} \) of ordinal numbers such that \( \alpha_n < \beta \) for each \( n \) and \( \alpha_n \to \beta \), i.e. for every \( \alpha < \beta \), there exists \( n \) such that \( \alpha < \alpha_n < \beta \). By dropping a finite number of terms if necessary, we may assume that \( \{x_m^{(\alpha_n)}\} \) is a subsequence of \( \{x_m^{(\alpha_p)}\} \) whenever \( p < n \). We then define \( \{x_m^{(\beta)}\} \) to be the sequence constructed from \( \{x_m^{(\alpha_n)}\} \) by the diagonal process, i.e., \( x_m^{(\beta)} = x_m^{(\alpha_n)} \). Then \( \{x_m^{(\beta)}\} \) is an essential subsequence of \( \{x_m^{(\alpha_n)}\} \) for each \( n \). Since \( \alpha_n \to \beta \), \( \{x_m^{(\beta)}\} \) is an essential subsequence of \( \{x_m^{(\alpha)}\} \) whenever \( \alpha < \beta \).

Hence \( \{x_m^{(\alpha)}\} \) are defined for all \( \alpha < \Omega \). Now for each \( \alpha < \Omega \), we let \( r_\alpha \) be the asymptotic radius of \( \{x_m^{(\alpha)}\} \) in \( C \). Since \( \{x_m^{(\alpha)}\} \) is an essential subsequence of \( \{x_m^{(\delta)}\} \) whenever \( \delta < \gamma \), and since \( r_\alpha \geq 0 \) for every \( \alpha < \Omega \), the transfinite sequence \( \{r_\alpha: \alpha < \Omega\} \) on the real line is decreasing and has lower bound \( 0 \). Let \( s = \inf\{r_\alpha: \alpha < \Omega\} \). Then clearly \( \lim\{r_\alpha: \alpha < \Omega\} \) exists and equals \( s \). This can happen only if for some \( \beta_0 < \Omega \), \( r_\alpha = s \) for all \( \alpha \) with \( \beta_0 < \alpha < \Omega \). Let
$\alpha$ be a fixed ordinal with $\beta_0 < \alpha < \Omega$. We shall show that the asymptotic center $z_\alpha$ of $\{x_\alpha\}$ is a fixed point of $T$.

From the way that $\{x^{(\alpha+1)}_m\}$ is constructed from $\{x^{(\alpha)}_m\}$, there exists a convergent sequence $\{p_m\} \subseteq Tz_\alpha$ with $\lim p_m = p \in Tz_\alpha$ such that

$$\|p_m - x^{(\alpha+1)}_m\| \leq \|z_\alpha - x^{(\alpha+1)}_m\|$$

for all $m$, where $y^{(\alpha+1)}_m = i(x^{(\alpha+1)}_m)$. Since $\{x^{(\alpha+1)}_m\}$ is a subsequence of $\{x^{(\alpha)}_m\}$, and $x^{(\alpha+1)}_m - y^{(\alpha+1)}_m \to 0$, we have from (1):

$$\limsup_m \|p - x^{(\alpha+1)}_m\| = \limsup_m \|p - y^{(\alpha+1)}_m\| \leq \limsup_m \|z_\alpha - x^{(\alpha+1)}_m\| \leq \limsup_m \|z_\alpha - x^{(\alpha)}_m\| = r_\alpha = r_{\alpha+1}.$$ 

It follows from the uniqueness of the asymptotic center that $p = z_{\alpha+1}$ and $z_\alpha = z_{\alpha+1}$, where $z_{\alpha+1}$ is the asymptotic center of $\{x^{(\alpha+1)}_m\}$ in $C$. Hence $z_\alpha = p \in Tz_\alpha$, completing the proof.

**Remark.** Theorem 1 remains true if $X$ is required only to be reflexive and uniformly convex in every direction [6], [3], since in such spaces the asymptotic center of a bounded sequence in a closed convex set is unique [10].

We do not know whether Theorem 1 is true when $C$ is required only to be weakly compact and to have normal structure. For the application of asymptotic center under this setting, see [8] and [9].

**References**


10. ———, *On asymptotic center and its applications to fixed point theory* (submitted).


DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

Current address: Department of Mathematics, University of Chicago, Chicago, Illinois 60637