A CONJECTURE OF M. GOLOMB ON OPTIMAL
AND NEARLY-OPTIMAL LINEAR APPROXIMATION

BY WOLFGANG DAHMEN$^1$ AND ERNST GORLICH

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In 1964, M. Golomb, in his survey paper on optimal and nearly-optimal linear approximation, presented at the General Motors Conference [3], called attention to an unsolved problem. It is the purpose of this note to solve this problem and at the same time to give a certain extension of the Haršiladze-Lozinskiï theorem.

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Let $C_{2\pi}$ be the space of continuous $2\pi$-periodic functions with Čebyšev norm, $\Pi_n$ the class of trigonometric polynomials of degree $\leq n$, and $E_n[f] = \inf\{ \|f - p\|; p \in \Pi_n\}$ the error of best approximation of an $f \in C_{2\pi}$ by elements of $\Pi_n$, for an $n \in P = \{0, 1, 2, \cdots\}$. A sequence $\{U_n\}_{n \in P}$ of bounded linear operators on $C_{2\pi}$ into $C_{2\pi}$ is called asymptotically optimal [3] for a given subset $Y \subset C_{2\pi}$ if

$$\sup_{f \in Y} \|f - U_n f\| \leq M_Y \sup_{f \in Y} E_n[f] \quad (n \in P),$$

$M_Y$ being some constant. $\{U_n\}$ is called optimal for $Y$ if (1) is satisfied with $M_Y = 1$.

In particular, $Y$ will be taken to be one of the spaces $C_0^r$, $r \in P$ or $A_0^\alpha$, $\alpha > 0$, where $C_0^r$ consists of those $f \in C_{2\pi}$ whose $r$th derivative is continuous and satisfies $\|f^{(r)}\| \leq 1$, and $A_0^\alpha$ is the class of functions $f(z)$ of a complex variable $z = x + iy$ which are $2\pi$-periodic in $x$, real for $y = 0$, analytic in the open strip $|y| < \alpha$, continuous in $|y| \leq \alpha$, and satisfy


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By the Jackson and Bernstein theorems, a sequence \( \{U_n\} \) of bounded linear operators is asymptotically optimal for some \( C_0^r \) [some \( A_0^\alpha \)] iff \( \|f - U_n f\| = O(n^{-r}) \) \( \{O(e^{-\alpha n})\} \), \( n \to \infty \), for all \( f \in C_0^r \) \( \{f \in A_0^\alpha \} \). Moreover, since \( \sup \{E_n[f]; f \in C_0^r\} = \mu_r(n + 1)^{-r} \) for all \( n \in P \), where \( \mu_r \) denote the Favard-Achieser-Kreïn constants \( (r \in P) \), a sequence \( \{U_n\} \) is optimal for some \( C_0^r \) iff \( \|f - U_n f\| \leq \mu_r(n + 1)^{-r} \) for all \( f \in C_0^r \), \( n \in P \).

Golomb's conjecture [3] consists of the following two statements.

(A) There does not exist a sequence \( \{U_n\} \) of bounded linear polynomial (i.e. \( U_n(C_{2\pi}) \subset \Pi_n \) for all \( n \in P \)) operators which is asymptotically optimal for all the classes \( C_0^r, \alpha > 0 \), and at the same time for all the classes \( A_0^\alpha, r \in P \).

(B) There does not exist a sequence of bounded linear polynomial operators which is optimal for all classes \( C_0^r, r \in P \).

In case (A), this was motivated by the fact that the Fourier partial sums \( S_n \) are asymptotically optimal for each \( A_0^\alpha, \alpha > 0 \), but not for any \( C_0^r \), \( r \in P \), whereas the de La Vallée Poussin sums \( V_n = (n - [n/2] + 1)^{-1} \cdot \sum_{k=[n/2]}^{n} S_k \) are asymptotically optimal for each \( C_0^r, r \in P \), but not for any \( A_0^\alpha, \alpha > 0 \). Concerning (B), for each class \( C_0^r \) there exists an optimal sequence of convolution type operators, but it depends on \( r \) and is unique at least among convolutions.

To prove (A) assume the contrary to be valid. If \( \{U_n\} \) is the sequence in question, define a sequence \( \{\overline{U}_n\} \) of bounded linear polynomial operators by

\[
(2) \quad \overline{U}_n f = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{-t} U_n T_t f \, dt, \quad T_t f(x) = f(x + t),
\]

according to Marcinkiewicz' device [5]. Then the \( \overline{U}_n \) are convolutions and they are asymptotically optimal for all \( C_0^r, A_0^\alpha, r \in P, \alpha > 0 \) as well. Thus the following two theorems may be applied in order to derive a contradiction.

**Theorem 1.** If \( \{U_n\} \) is a sequence of bounded linear polynomial operators on \( C_{2\pi} \) which is asymptotically optimal for some \( A_0^\alpha, \alpha > 0 \), then \( \lim \sup_{n \to \infty} \|U_n\| = +\infty \).

**Theorem 2.** If \( \{U_n\} \) is a sequence of bounded linear polynomial convolution operators on \( C_{2\pi} \) which is asymptotically optimal for some \( C_0^r \), \( r \in P \), then \( \|U_n\| = O(1), n \to \infty \).
The proof of Theorem 1 proceeds via (2) and makes use of a weak version of an inequality of Hardy-Littlewood [4] and Sidon [8] (to be found e.g. in Nikol'skii [6, p. 262]). Theorem 2 is proved by an application of Bernstein's inequality to \((U_n - V_n)f\).

For the proof of (B) assume that \(\{U_n\}\) satisfies \(\|f - U_n f\| \leq \mu_r(n + 1)^{-r}\) for all \(f \in C_0^r, n, r \in \mathbb{P}\). Then the following Lemma furnishes a contradiction to the fact that the \(\mu_r\) are bounded uniformly in \(r\).

**LEMMA.** If \(\{U_n\}\) is a sequence of bounded linear polynomial operators on \(C_{2\pi}\) such that for each \(r \in \mathbb{P}\)

\[
\sup_{f \in C_0^r} \|f - U_n f\| \leq M_r(n + 1)^{-r} \quad (f \in C_0^r, n \in \mathbb{P}),
\]

then \(\limsup_{r \to \infty} M_r = +\infty\).

This is a consequence of (2) and of the inequality mentioned above (see [8]).

In this context let us mention the familiar Haršiladze-Lozinskii theorem (see e.g. [2, pp. 212, 233]) which asserts that there does not exist a sequence \(\{U_n\}\) of bounded linear polynomial operators satisfying simultaneously

(a) \(U_n(U_n f) = U_n f\) for each \(n \in \mathbb{P}, f \in C_{2\pi}\), and

(b) \(\|f - U_n f\| \rightarrow 0\) as \(n \rightarrow \infty\) for each \(f \in C_{2\pi}\).

Extensions of this result have been given e.g. by Berman [1] and Sapogov [7]. As a consequence of the above, another extension is obtained on replacing the projection condition (a) by (a') or (a'') below.

(a') \(\{U_n\}\) is asymptotically optimal for some \(A_0^\alpha, \alpha > 0\).

(a'') \(\{U_n\}\) satisfies (3) for each \(r \in \mathbb{P}\), and \(M_r = O(1), r \rightarrow \infty\).

Details will appear elsewhere.

**REFERENCES**


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LEHRSTUHL A FÜR MATHEMATIK, TECHNOLOGICAL UNIVERSITY, 51 AACHEN, FEDERAL REPUBLIC OF GERMANY