ON SPACES OF RIEMANN SURFACES WITH NODES

BY LIPMAN BERS

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This is a summary of results, to be published in full elsewhere, which strengthen and refine the statements made in a previous announcement [1].

A compact Riemann surface with nodes of (arithmetic) genus $p > 1$ is a connected complex space $S$, on which there are $k = k(S) > 0$ points $P_1, \ldots, P_k$, called nodes, such that (i) every node $P_j$ has a neighborhood isomorphic to the analytic set $\{z_1z_2 = 0, |z_1| < 1, |z_2| < 1\}$, with $P_j$ corresponding to $(0, 0)$; (ii) the set $S \setminus \{P_1, \ldots, P_k\}$ has $r > 1$ components $\Sigma_1, \ldots, \Sigma_r$, called parts of $S$, each $\Sigma_i$ is a Riemann surface of some genus $p_i$, compact except for $n_i$ punctures, with $3p_i - 3 + n_i > 0$, and $n_1 + \cdots + n_r = 2k$; and (iii) we have

$$p = (p_1 - 1) + \cdots + (p_r - 1) + k + 1.$$

Condition (ii) implies that every part carries a Poincaré metric, and condition (iii) is equivalent to the requirement that the total Poincaré area of $S$ be $4\pi(p - 1)$.

From now on $p$ is kept fixed and the letter $S$, with or without subscripts or superscripts, always denotes a surface with properties (i)–(iii). If $k(S) = 0$, $S$ is called nonsingular; if $k(S) = 3p - 3$, $S$ is called terminal.

A continuous surjection $f: S' \rightarrow S$ is called a deformation if for every node $P \in S$, $f^{-1}(P)$ is either a node or a Jordan curve avoiding all nodes and, for every part $\Sigma$ of $S$, $f^{-1}\Sigma$ is an orientation preserving homeomorphism. Two deformations, $f: S' \rightarrow S$ and $g: S'' \rightarrow S$ are called equivalent if there are homeomorphisms $\varphi: S' \rightarrow S''$ and $\psi: S \rightarrow S$, homotopic to an isomorphism and to the identity, respectively, such that $g \circ \varphi = \psi \circ f$. The deformation space $D(S)$ consists of all equivalence classes $[f]$ of deformations onto $S$. To every node $P \in S$ belongs a distinguished subset consisting
of all \([f] \in D(S)\) with \(f^{-1}(P)\) a node of \(f^{-1}(S)\).

We define a Hausdorff topology on \(D(S)\) as follows. If \(c\) is a closed curve on a part of \(S\), denote by \(|c|\) the length of the unique geodesic freely homotopic to \(S\). Let \(C\) be a finite set of closed curves on parts of \(S\), \(\epsilon\) a positive number, and \(\omega: S' \to S\) a deformation. We say that \(\omega\) is \((C, \epsilon)\) small if for every Jordan curve \(c'\) on a part of \(S'\) such that \(\omega(c')\) is a node, \(|c'| < \epsilon\), and for every \(c \in C, |\omega^{-1}(c)| - |c| < \epsilon\). A set \(A \subset D(S)\) is called open if, for every \([f] \in A\), there is a finite set \(C\) of closed curves on parts of \(f^{-1}(S)\), and a number \(\epsilon > 0\), such that whenever \(\omega: S' \to f^{-1}(S)\) is \((C, \epsilon)\) small, \([f \circ \omega] \in A\).

**Theorem 1.** \(D(S)\) is a cell. There is an (essentially canonical) homeomorphism of \(D(S)\) onto \(C^{3p-3}\) which takes each distinguished subset onto a coordinate hyperplane.

A deformation \(h: S \to S_0\) induces a mapping \(h^*: D(S) \to D(S_0)\), called an allowable mapping, which takes each \([f] \in D(S)\) into \([h \circ f]\).

**Theorem 2.** Let \(S\) and \(S_0\) have the same genus, and let \(k(S_0) = k(S) + l\). If \(l = 0\), an allowable mapping \(D(S) \to D(S_0)\) is a homeomorphic bijection. If \(l > 0\), an allowable mapping \(D(S) \to D(S_0)\) is a universal covering of the complement of \(l\) distinguished subsets.

The proofs of Theorems 1 and 2 use the so-called Fenchel-Nielsen coordinates (cf. [1, p. 51]). An inequality for Fenchel-Nielsen coordinates stated in [1] as Theorem XV (and previously conjectured by Mumford) implies

**Theorem 3.** Let \(S_1, \ldots, S_m\) be all not isomorphic terminal surfaces of genus \(p\). There are compact sets \(K_j \subset D(S), j = 1, \ldots, m\), such that every \(S\) is of the form \(S = f^{-1}(S_j), [f] \in K_j\) for some \(j\).

If \(S\) is nonsingular, \(D(S)\) can be identified with the Teichmüller space \(T_p\) of closed Riemann surfaces of genus \(p\). For every \(S\), each point in \(D(S)\), not belonging to a distinguished subset, has a neighborhood which can be naturally identified with a neighborhood in \(T_p\). Thus an open dense set in \(D(S)\) is a complex manifold. It follows that \(D(S)\) has the structure of a ringed space.

**Theorem 4.** \(D(S)\) is a complex manifold which can be realized as a bounded domain in \(C^{3p-3}\). The distinguished subsets of \(D(S)\) are nonsingular analytic hypersurfaces which meet transversally.
The proof utilizes the Kleinian groups constructed in [1, pp. 46–47]. The spaces $X_a(S)$ used there are finite ramified coverings of $D(S)$. The following statement is almost obvious.

**Theorem 5.** Allowable mappings are holomorphic.

Let $\Gamma(S)$ be the group of allowable self-mappings of $D(S)$ induced by all topological orientation preserving self-mappings of $S$, and let $\Gamma_0(S)$ be the subgroup induced by the automorphisms (conformal self-mappings) of $S$. Note that if $\gamma([f]) = [g]$ for some $\gamma \in \Gamma(S)$, then $f^{-1}(S)$ is isomorphic to $g^{-1}(S)$. The converse statement is, in general, false.

**Theorem 6.** The group $\Gamma(S)$ is discrete, the subgroup $\Gamma_0(S)$ is finite and is the stabilizer of $[\text{id}] \in D(S)$ in $\Gamma(S)$.

Let $M_p$ denote the moduli space (Riemann space) for genus $p$, that is, the set of all isomorphism classes $[S]$ of Riemann surfaces with nodes, of genus $p$. We define a Hausdorff topology in $M_p$ by calling a set $B \subset M_p$ open if, for every $[S] \in B$, there is a finite set $C$ of closed curves on parts of $S$, and an $\epsilon > 0$, such that $[S'] \in B$ whenever there is a $(C, \epsilon)$ small deformation $S' \to S$. The moduli space of nonsingular Riemann surfaces of genus $p$ is known to be a complex space, and is an open dense subset of $M_p$. Hence $M_p$ has the structure of a ringed space.

There is a canonical mapping $D(S) \to M_p$ which sends $[f] \in D(S)$ into $[f^{-1}(S)]$.

**Theorem 7.** The canonical mapping $D(S) \to M_p$ is holomorphic. Furthermore, $[\text{id}] \in D(S)$ has a neighborhood $N$, stable under $\Gamma_0(S)$, such that $N/\Gamma_0(S)$ is isomorphic to a neighborhood of $[S]$ in $M_p$.

Theorems 3 and 7 imply the known (cf. [2])

**Corollary (Mayer-Mumford).** $M_p$ is a compact normal complex space (and a $V$-manifold).

A regular $q$-differential on $S$ is defined by assigning a holomorphic form $F_\Sigma$ of type $(q, 0)$ to each part $\Sigma$ of $S$; the $F_\Sigma$ should be either regular at the punctures, or have there poles of order not exceeding $q$, the "residues" at two punctures joined in a node being equal (if $q$ is even) or opposite (if $q$ is odd). The number $\delta(p, q)$ of linearly independent regular $q$-differentials is $p$ if $q = 1$, $(2q - 1)(p - 1)$ if $q > 1$. If we choose $\delta = \delta(p, q)$ linearly independent $q$-differentials, their "values" at every point of...
$S$, including a node, are the homogeneous coordinates of a point in $\mathbb{P}_{5,-1}$.
In this way one obtains a holomorphic mapping $S \to \mathbb{P}_{5,-1}$, the so-called $q$-canonical mapping. This is an embedding for $q > 2$ and, in some cases, also for $q = 2$ and $q = 1$.

**Theorem 8.** For every $S$ and every $q \geq 1$, there is an analytic hypersurface $\sigma \subset D(S)$, with $[id] \notin \sigma$, and a holomorphic mapping $\Phi$ of $D(S) \setminus \sigma$ into the Chow variety of curves of degree $2q(p - 1)$ in $\mathbb{P}_{5(p,q)-1}$ such that, for $[f] \in D(S) \setminus \sigma, \Phi([f])$ is the Chow point of a $q$-canonical image of $f^{-1}(S).

The proof uses the Poincaré series described in [1, pp. 48–49]. If $S$ is nonsingular, one knows, from other considerations, that the result is true with $\sigma = \emptyset$. For singular $S$, I could thus far obtain that $\sigma = \emptyset$ only for $q = 1$.

**References**


DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027