DIFFERENTIAL INEQUALITIES AND CARATHÉODORY FUNCTIONS

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ABSTRACT. The author proves a very general result from which it is possible to show that a regular function satisfying a differential inequality of a certain type is necessarily a Carathéodory function. This result has applications in the theory of univalent functions.

Let \( \mathcal{P} \) denote the class of Carathéodory functions; that is, functions \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) regular in the unit disc \( \Delta \), and for which \( \text{Re} \, p(z) > 0 \).

In a recent paper [2] it was shown that if \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) is regular in \( \Delta \), with \( p(z) \neq 0 \) in \( \Delta \), and if \( \alpha \) is a real number, then for \( z \in \Delta \)

\[
\text{Re} \left[ p(z) + \alpha (zp'(z)/p(z)) \right] > 0 \Rightarrow \text{Re} \, p(z) > 0;
\]

that is, \( p(z) \in \mathcal{P} \).

In this note we replace the differential inequality in (1) by a much more general condition which will still imply that \( p(z) \) is a Carathéodory function.

DEFINITION 1. Let \( u = u_1 + u_2 i \) and \( v = v_1 + v_2 i \), and let \( \Psi \) be the set of functions \( \psi(u, v) \) satisfying:

(a) \( \psi(u, v) \) is continuous in a domain \( D \) of \( C \times C \);
(b) \( (1, 0) \in D \) and \( \text{Re} \, \psi(1, 0) > 0 \);
(c) \( \text{Re} \, \psi(u_2 i, v_2) \leq 0 \) when \( (u_2 i, v_2) \in D \) and \( v_2 \leq -1/2(1 + u_2^2) \).

We denote by \( \Phi \) the subset of \( \Psi \) which satisfies (a), (b) and the following condition:

(c') \( \text{Re} \, \psi(u_2 i, v_2) \leq 0 \) when \( (u_2 i, v_2) \in D \) and \( v_2 \leq 0 \).

EXAMPLES. It is easy to check that each of the following functions are in \( \Psi \).

\[
\phi_1(u, v) = u + \alpha v/u, \quad \alpha \text{ real, with } D = [C - \{0\}] \times C. \\
\phi_2(u, v) = u^2 + v \text{ with } D = C \times C. \\
\phi_3(u, v) = u + \alpha v, \quad \alpha \geq 0, \text{ with } D = C \times C. \\
\phi_4(u, v) = u - v/u^2 \text{ with } D = [C - \{0\}] \times C. \\
\phi_5(u, v) = -\ln(1/2 - v) \text{ with } D = C \times \{(v_1, v_2) | v_1 < 1/2\}.
\]

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Note that $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ are also in $\Phi$ but $\psi_5 \notin \Phi$. The set $\Phi$ is thus a proper subset of $\Psi$. Though some generality is lost in considering the class $\Phi$ as opposed to considering $\Psi$, the former is much easier to work with algebraically.

**Definition 2.** Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be regular in $\Delta$ and let $\psi \in \Psi$ with corresponding domain $D$. We denote by $\mathcal{P}(\psi)$ those functions $p(z)$ that satisfy:

(i) $(p(z), zp'(z)) \in D$, and

(ii) $\text{Re} \psi(p(z), zp'(z)) > 0$,

when $z \in \Delta$.

Note that $\mathcal{P}(\psi)$ is not empty, since for all $\psi \in \Psi$ it is true that $p(z) = 1 + p_1z \in \mathcal{P}(\psi)$ for $p_1$ sufficiently small (depending on $\psi$). It appears further that most $\psi \in \Psi$ provide a large number of other functions in $\mathcal{P}(\psi)$.

Our main result is the following theorem.

**Theorem 1.** For any $\psi \in \Psi$, $\mathcal{P}(\psi) \subset \mathcal{P}$.

In other words the Theorem states that if $\psi \in \Psi$, with corresponding domain $D$, and if $(p, zp') \in D$ then

(2) $\text{Re} \psi(p(z), zp'(z)) > 0 \Rightarrow \text{Re} p(z) > 0$.

Since $\Phi \subset \Psi$, we immediately have the following Corollary.

**Corollary.** For any $\psi \in \Phi$, $\mathcal{P}(\psi) \subset \mathcal{P}$.

The proof of the Theorem is involved and will not be presented here. However an independent proof of the Corollary follows.

Let $p(z) \in \mathcal{P}(\psi)$, and assume there exists a point $z_0 = r_0 \exp(i\theta_0) \in \Delta$ such that $\text{Re} p(z) \geq 0$ for $|z| \leq r_0$, and $\text{Re} p(z_0) = 0$. Thus $p(z_0) = ai$, where $a$ is a real number. We now show that $z_0p'(z_0) = k$, where $k \leq 0$. Since the result is true if $p'(z_0) = 0$, we need only consider the case $p'(z_0) \neq 0$. The curve $p(r_0e^{it})$ is tangent to the imaginary axis at $z_0$, and so we have $\arg z_0p'(z_0) = \pi$; that is $z_0p'(z_0) = k$, where $k < 0$. Hence at $z_0$ we have $\text{Re} \psi(p, zp') = \text{Re} \psi(ai, k)$ with $a$ real and $k \leq 0$. But this implies that $\text{Re} \psi(p, zp') \leq 0$ at $z = z_0$, which is a contradiction of the fact that $p(z) \in \mathcal{P}(\psi)$. Hence $\text{Re} p(z) > 0$ for $z \in \Delta$.

**Remarks.** If we apply the Theorem (or the Corollary) to the example $\psi_1(u, v)$, we obtain condition (1). Applying it to $\psi_2$, $\psi_3$ and $\psi_4$ we obtain respectively:

(3) $\text{Re}[p^2(z) + zp'(z)] > 0 \Rightarrow \text{Re} p(z) > 0$;

(4) $\text{Re}[p(z) + azp'(z)] > 0$, with $a \geq 0 \Rightarrow \text{Re} p(z) > 0$,

and

(5) $p(z) \neq 0$ and $\text{Re}[p(z) - zp'(z)p(z)] > 0 \Rightarrow \text{Re} p(z) > 0$. 

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We see that for different \( \psi \in \Psi \) we can obtain different differential conditions for \( p(z) \) to be a Carathéodory function. By appropriately choosing \( \psi \in \Psi \) we can define many new subclasses of \( \mathcal{P} \) and can prove many properties of the class \( \mathcal{P} \).

The theorem has many applications in the theory of univalent functions. If we set \( p(z) = zf'(z)/f(z) \) in Theorem 1, we see from (2) that each \( \psi \in \Psi \) generates a subclass of starlike functions. In particular \( \psi_1(u, v) = u + \alpha v/u \) generates the class of alpha-convex functions [2]. Similarly by setting \( p(z) = e^{i\gamma}z f'(z)/f(z) \), where \( |\gamma| < \frac{1}{2} \), or \( p(z) = f'(z)/g'(z) \), where \( g(z) \) is convex, and using slightly modified forms of Definitions 1 and 2 and Theorem 1, we can generate many new subclasses of spiral-like and close-to-convex functions, respectively. These results, the proof of Theorem 1, and other applications will appear in a forthcoming paper [1].

REFERENCES


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