COMMUTATIVE SUBALGEBRA OF $L^1(G)$ ASSOCIATED
WITH A SUBELLiptic OPERATOR ON A LIE GROUP $G$

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1. Introduction. Let $G$ be a Lie group and $LG$ its Lie algebra regarded as the space of differential operators of the first order which commute with the right translations. If $X_1, \ldots, X_n$ is a basis of $LG$, then the operator $L = X_1^2 + \cdots + X_n^2$ is called a laplacian on $G$. In [4] the commutative Banach $\ast$-subalgebra of $L^1(G)$ generated by the fundamental solution of the heat equation $(\partial/\partial t - L)u = 0$ was studied, and in case of compact extensions of nilpotent groups it proved to be useful in studying spectral properties of $L$ on various $L^p(G)$ spaces, as well as in proving tauberian Wiener theorems concerning Gauss and Poisson integrals. In [6] and [9] a powerful method of singular integrals on the class of nilpotent Lie groups admitting one-parameter groups of dialations was developed. In [1] and [2] Folland and Stein studied the relation of these to certain subelliptic operators on the Heisenberg group. The idea is that in various important cases, although for a given one-parameter group of dialations $\{\delta_s\}$, $s > 0$, of $G$ there is no basis in $LG$ such that $\delta_s \ast L = s \lambda L$ where $\lambda$ is a scalar, there exists a set of generators $X_1, \ldots, X_k$ of the Lie algebra $LG$ such that $\delta_s \ast X_j = s X_j$, $j = 1, \ldots, k$. Let
\begin{equation}
L = X_1^2 + \cdots + X_k^2.
\end{equation}
Then, of course,
\begin{equation}
\delta_s \ast L = s^2 L.
\end{equation}
The fact that $X_1, \ldots, X_k$ generate $LG$ as a Lie algebra implies that $L$ is a subelliptic operator. Using this fact we shall construct the Gauss and Poisson kernels for the operator $L$, and via a study of the subalgebra of $L^1(G)$ generated by these, we obtain the equality of the spectra of $L$ on various $L^p(G)$ spaces as well as the corresponding tauberian Wiener theorems. More-


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over (2) implies very natural transformation rules for the Gauss and Poisson kernels under the dialations.

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2. Theorems. Consider $L$ as a densely defined symmetric operator on $L^2(G)$. Let $L' = L$ be its adjoint. Then, as it follows from [3], $L$ is a selfadjoint nonpositive definite operator.

From [7] and [8] we deduce the following version of

**Sobolev's Lemma.** There exists an $N$ such that for every compact set $Ω$ in $G$ there exists a constant $C$ such that $|f(x)| \leq C\sum_{j=0}^{N} |\langle f, \phi_j \rangle|$ for all $x$ in $Ω$ and $f$ in $\mathcal{D}(1,N)$.

Now we are able to define the Gauss and Poisson kernel pretty much the same way as in [4] and [11].

**THEOREM 1.** There exists a unique one-parameter semigroup $\{p_t\}_{t>0}$ of (i) nonnegative, (ii) normalized functions in $L^p(G)$ such that

(iii) $p_{s+t} = p_s p_t$ for every $f$ in $L^p(G)$,

(iv) $\lim_{t \to 0} |p_t f|_p = 0$, $1 \leq p < \infty$,

(v) the function $(0, \infty) \times G \ni t, x \to p_t(x)e^x$ is $C^\infty$, and if $u(t, x) = p_t \ast f(x)$,

(4) \[ \frac{d}{dt} - L)u(t, x) = 0 \]

for all $f \in L^p(G)$, $1 \leq p \leq \infty$.

Let

(5) \[ p^r(x) = (\pi)^{-\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}} e^{-\lambda} p_{t^2/4\lambda} d\lambda. \]

Then $\{p^r\}_{r>0}$ is a semigroup of functions in $L^1(G)$ such that (i)--(iv) are satisfied, and if $u(t, x) = p^r \ast f$, then

(6) \[ (d^2/dt^2 + L)u(t, x) = 0 \]

for all $f \in L^p(G)$, $1 \leq p \leq \infty$.

Understandably enough, the function $u(t, x) = p_t \ast f(x)$ is called the Gauss integral of $f$, and the function $v(t, x) = p^r \ast f(x)$ is called the Poisson integral of $f$.

We have also the following version of Nelson’s lemma [10].
Theorem 2. For every nonnegative submultiplicative function \( \phi \) on \( G \) and every \( t_0 > 0 \), there is a constant \( C \) such that \( \int p_t(x) \phi(x) \, dx < C \) for all \( t < t_0 \).

Let \( A = \text{lin} \{ p_t : t > 0 \} \). Then \( A \) is a commutative *-subalgebra of \( L^1(G) \). Let \( A \) denote its closure in the \( L^1 \) norm. Then of course, \( p^t \in A \).

From now on we assume that the group \( G \) is of polynomial growth (e.g. a compact extension of a nilpotent group; cf. [5]).

Let \( \text{Sp}_p = \{ \lambda \in \mathbb{C} : (\lambda - L)^{-1} \text{ is a bounded operator on } L^p(G) \} \).

Theorem 3. \( A \) is symmetric, i.e. \( \text{Sp}_p f * f^* \) is a real nonnegative for all \( f \) in \( A \), hence \( \text{Sp}_p L = \text{Sp}_2 L \) for all \( 1 < p < \infty \).

Theorem 4. The Gelfand space of \( A \) is naturally homeomorphic with \( \text{Sp}_2 L \).

Theorem 5. There is an integer \( r \) depending on the group only such that the functions \( F \in C'_c(\mathbb{R}) \) operate on the hermitian functions \( f \) in \( A \) into \( A \).

Theorem 6. The algebra \( A \) is regular and the set of functions \( f \) in \( A \), such that \( \text{supp} \, \hat{f} \) is compact, is dense in \( A \). Hence

(i) Every proper ideal of \( A \) is annihilated by a nonzero homomorphism of \( A \) into \( C \).

(ii) None of the \( p_t \)'s and \( P^t \)'s, \( t > 0 \), is contained in a proper left (or right) ideal of \( L^1(G) \).

(iii) If \( u(t, x) \) is a solution of (4) or (6) such that \( u(0, x) = \phi(x) \) and \( |u(t, \cdot)|_\infty \leq C, t > 0 \), then, if for a \( t_0 > 0 \), \( \lim_{x \to \infty} u(t_0, x) = a \), then \( \lim_{x \to \infty} f * \phi(x) = a f \).

Now let \( G \) be a connected, simply-connected nilpotent Lie group and let \( \{ \delta_s \}_{s > 0} \) be a one-parameter group of dialations of \( G \). Suppose that \( X_1, \ldots, X_k \) generate \( LG \) as a Lie algebra and \( \delta_s \cdot X_j = s X_j \). For important examples where such a situation occurs see [1], [2], [6], [9], [12]. Let \( d(\delta_s x) = s^r dx \), and let \( L = X_1^2 + \cdots + X_k^2 \). We then have

(7) \( \delta_s \cdot L = s^2 L \).

From (7) we can easily deduce the following formulae

\[ p_t(x) = t^{-r/2} p(\delta_{t^{-1}}(x)) \quad \text{and} \quad P^s(x) = t^{-r} P^1(\delta_{t^{-1}}(x)). \]

Clearly enough, also the whole algebra \( A \) is stable under automorphisms \( \delta_s^* \), \( s > 0 \).

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The connection of $p_t$ and $P^t$ to the homogeneous norm functions (cf. [6]) and singular integrals on $G$ we hope to study in a subsequent paper. Details and proofs will appear elsewhere.

REFERENCES


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