A CONSTRUCTIVE CHARACTERIZATION
OF DISCONJUGACY

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An ordinary linear differential operator $L$ defined by

$$L y = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y$$

is said to be disconjugate on an interval $I$ if every nontrivial solution of

$$Ly = 0$$

has less than $n$ zeros on $I$, multiple zeros being counted according to their multiplicity.

We assume $a_i \in C(I)$ for $i = 0, \cdots, n-1$. This assumption is made mainly for convenience and can be considerably weakened.

We announce here an algorithm for the construction of disconjugate operators of type (1) for any $n \geq 2$ and all intervals $I$. Our construction yields all disconjugate operators of type (1) if the interval $I$ is either open or compact. This construction has the following features: It is iterative or inductive in the sense that the set of $n$th order disconjugate operators is constructed from the set of $(n-1)$st order ones. (The second order from the first order ones. All first order operators $y' + a_0(t)y$ are disconjugate.)

The procedure for going from $n-1$ to $n$ involves a parameter function.

For the remainder of this paper $I$ denotes any compact or open interval, $C(I)$, the set of real valued continuous functions on $I$ and $C'(I)$ the set of real valued functions on $I$ which have continuous first derivatives.

**Theorem 1.** Given $a_0$ in $C(I)$ there exist $a_1, \cdots, a_{n-1}$ in $C(I)$ such that (1) is disconjugate. Moreover (1) is disconjugate if and only if there exists $r$ in $C(I)$ and $b_0, \cdots, b_{n-2}$ in $C'(I)$ such that:

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From Theorem 1 the following algorithm for the construction of $n$th order disconjugate operators of type (1) is obtained: Start with $a_0$ in $C(I)$. Choose any $r$ in $C(I)$. Let $b_0$ be any solution of $x' + rx = a_0$, i.e.,

$$b_0(t) = \exp \left( - \int_{t_0}^t r(s) ds \right) \left[ c + \int_{t_0}^t \exp \left( \int_{t_0}^u r(s) ds \right) a_0(u) du \right]$$

for any $t_0$ in $I$ and constant $c$. Choose $b_1, \ldots, b_{n-2}$ such that (i) holds. Determine $a_i$ for $i = 1, \ldots, n-1$ by the second and third equations under (ii). Then the operator $L$ determined by (1) is disconjugate. Furthermore all disconjugate operators $L$ of type (1) on open or compact intervals are obtained this way.

For purposes of illustration we discuss this construction for the cases $n = 2$ and $n = 3$. The case $n = 2$: $Ly = y'' + a_1 y' + a_0 y$. The idea is to start with a given $a_0$ in $C(I)$ and characterize all functions $a_1$ for which $L$ is disconjugate. The characterization is

$$a_1(t) = r(t) + \exp \left( - \int_{t_0}^t r(s) ds \right) \left[ c + \int_{t_0}^t \exp \left( \int_{t_0}^u r(s) ds \right) a_0(u) du \right]$$

where $t_0$ is any point in $I$, $c$ is an arbitrary constant, and $r$ is an arbitrary function in $C(I)$.

In the literature, disconjugacy of second order equations is usually discussed for equations in the form $y'' + qy$ or $(ry')' + qy$. Consider $y'' + qy$. The question of disconjugacy for this equation reduces to: When is $a_1$ in (3) (with $a_0 = q$) zero? Setting $a_1(t) \equiv 0$ and differentiating reduces (3) to $r' + r^2 + a_0 \equiv 0$. This is the famous Riccati equation associated with the form $y'' + a_0 y$. So our characterization (3)—in the second order case for the special form $y'' + a_0 y$—reduces to the well-known equivalence between disconjugacy and existence of solutions of the Riccati equation.

The case $n = 3$: $Ly = y''' + a_2 y'' + a_1 y' + a_0 y$. Again the idea is to start with any function $a_0$ and determine all pairs of functions $(a_1, a_2)$ which make $L$ disconjugate. This is done as follows: Let $r$ be in $C(I)$ and let $b_0$ be any solution of $x' + rx = a_0$. Take any $b_1$ for which
$y'' + b_1 y + b_0 y$ is disconjugate (i.e., determine $b_1$ as discussed in the case $n = 2$ above). Now let $a_1 = b_1' + b_0 + rb_1$ and $a_2 = r + b_1$. Then $L$ is disconjugate and all third order disconjugate operators $L$ of type (1) on open or compact intervals are obtained this way.

The proof of Theorem 1 is based on the following idea: The $n$th order operators (1) which are disconjugate are precisely those determined by products

$$(y' + ry)(y^{(n-1)} + b_{n-2}y^{(n-2)} + \cdots + b_0y)$$

where the $(n - 1)$st order operator with coefficients $b_i$ is disconjugate. Product here is meant in the sense of composition. A detailed proof and related matters will be published elsewhere.

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